Formal Non-linear Optimization via Templates and Sum-of-Squares

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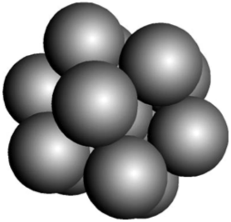
Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is $\dfrac{\pi}{\sqrt{18}}$

- It corresponds to the way people would intuitively stack oranges, as a pyramid shape
- The proof of T. Hales (1998) consists of thousands of non-linear inequalities
- Many recent efforts have been done to give a formal proof of these inequalities: Flyspeck Project
- Motivation: get positivity certificates and check them with Proof assistants like Coq



Inequalities issued from Flyspeck non-linear part involve:

1. Multivariate Polynomials:
$x_1x_4(-x_1+x_2+x_3-x_4+x_5+x_6)+x_2x_5(x_1-x_2+x_3+x_4-x_5+x_6)+ x_3x_6(x_1+x_2-x_3+x_4+x_5-x_6)-x_2(x_3x_4+x_1x_6)-x_5(x_1x_3+x_4x_6)$
2. Semi-Algebraic functions algebra $\mathcal{A}$: composition of polynomials with $|\cdot|, \sqrt{}, +, -, \times, /, \sup, \inf, \cdots$
3. Transcendental functions $\mathcal{T}$: composition of semi-algebraic functions with $\boldsymbol{arctan}$, $\boldsymbol{exp}$, $+, -, \times, \cdots$

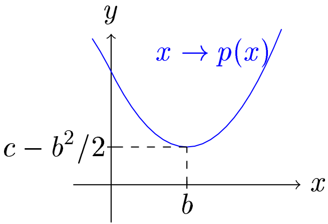
Lemma from Flyspeck (inequality ID 6096597438)

$\forall x \in [3, 64], 2\pi - 2(x \arcsin(\cos(0.797)\sin(\pi/x)) - (0.591 - 0.0331x + 1.506) \geq 0$

Certification: who does what?

Polynomial Optimization (POP): $\min_{x \in \mathbb{R}} p(x) = 1/2x^2 - bx + c$

1. A program written in OCaml/C provides the Sum-of-Squares decomposition: $1/2(x-b)^2$
2. A program written in Coq checks: $\forall x \in \mathbb{R}, p(x) = 1/2(x-b)^2 + c - b^2/2$



- Sceptical approach: obtain *certificates* of positivity with efficient oracles and check them formally
- Questions: How to obtain the certificates? How to deal with non-polynomial case?