Formal Non-linear Optimization via Templates and Sum-of-Squares

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Motivation: Flyspeck-Like Problems
The Kepler Conjecture

Kepler Conjecture (1611):
The maximal density of sphere packings in 3D-space is $\frac{\pi}{18}$

- It corresponds to the way people would intuitively stack oranges, as a pyramid shape
- The proof of T. Hales (1998) consists of thousands of non-linear inequalities
- Many recent efforts have been done to give a formal proof of these inequalities: Flyspeck Project
- **Motivation:** get positivity certificates and check them with Proof assistants like Coq
Inequalities issued from Flyspeck non-linear part involve:

1. **Multivariate Polynomials:**
   \[
   x_1 x_4 (-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2 x_5 (x_1 - x_2 + x_3 + x_4 - x_5 + x_6) +
   x_3 x_6 (x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2 (x_3 x_4 + x_1 x_6) - x_5 (x_1 x_3 + x_4 x_6)
   \]

2. **Semi-Algebraic functions algebra \( \mathcal{A} \):** composition of polynomials with \(| \cdot |, \sqrt{\quad}, +, −, ×, /, sup, inf, \cdots \)

3. **Transcendental functions \( \mathcal{T} \):** composition of semi-algebraic functions with \( \arctan, \exp, +, −, ×, \cdots \)

**Lemma from Flyspeck (inequality ID 6096597438)**

\[
\forall x \in [3, 64], 2\pi - 2(x \arcsin(\cos(0.797) \sin(\pi/x)) - (0.591 - 0.0331x + 1.506) \geq 0
\]
Polynomial Optimization (POP): $\min_{x \in \mathbb{R}} p(x) = 1/2x^2 - bx + c$

1. A program written in OCaml/C provides the Sum-of-Squares decomposition: $1/2(x - b)^2$

2. A program written in Coq checks:
   $\forall x \in \mathbb{R}, p(x) = 1/2(x-b)^2 + c - b^2/2$

- Sceptical approach: obtain *certificates* of positivity with efficient oracles and check them formally
- Questions: How to obtain the certificates? How to deal with non-polynomial case?