Cylindrical Algebraic Decomposition in Coq

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- We are interested in intuitionistic the theory of discrete real closed fields (we will see it is the same as the classical).
- We have stated the theorem of quantifier elimination for the theory of discrete real closed fields.
- We want to formalize a (constructive) proof of this theorem in the Coq system, meaning we want a quantifier elimination program.

Formal definition of the field signature

Terms on the language of fields.

```
Inductive term : Type :=
| Var of nat
| C0 : term
| C1 : term
| Add of term & term
| Opp of term
| Mul of term & term
| Inv of term.
```

Formal definition of a first order theory

For an arbitrary type `term` of terms, formulas are:

```
Inductive formula (term : Type) : Type :=
| Equal of term & term
| Leq of term & term
| Not of formula
| And of formula & formula
| Or of formula & formula
| Implies of formula & formula
| Exists of nat & formula
| Forall of nat & formula.
```

Theory of real closed fields

We use a record type to define a type which is simultaneously equipped with a field signature and a theory of real closed fields.

```
Record rcf := RealClosedField{
  carrier : Type;
  Req : carrier -> carrier -> bool;
  Rleq : carrier -> carrier -> bool;
  zero : carrier;
  one : carrier
  opp : carrier -> carrier;
  add : carrier -> carrier -> carrier;
  mul : carrier -> carrier -> carrier;
  inv : carrier -> carrier;
  _ : associative add;
  _ : commutative add;
  _ : left_id zero add;
  _ : left_inverse zero opp add;
  ...}.
```

Instances of the theory of real closed fields

An instance of this theory is constructed when:

- We have formed a concrete type
  for instance the type `Ralg` of real algebraic numbers
- We have defined field constants and implemented field operations
  Zero, one, addition, ...
- We have proved the theorems specifying these operations
  Addition is commutative, ...
- We have gathered all this in an element of the record type

```
Definition Ralg_rcf :=
     RealClosedField Ralg Ralg0 Ralg1 Ralg_opp ...
```

What do we formalize?

- A signature Σ (of rings)

  The type `term`
- The terms on Σ

  The elements `t : term`
- The first order statements $\mathcal{F}(\Sigma, \mathbb{N})$

  The elements `f : formula`
- The definition of Σ-structure

  The type `rcf` (which contains specifications).
- The Σ-structures themselves

  The elements `MyRcf : rcf`

What do we formalize?

- An interpretation function $[t(\mathbf{x})]_{R,e}$ from terms in $\mathcal{L}(\Sigma, \mathbb{N}))$ in a Σ-structure

  `eval: (seq (carrier R))-> term -> (carrier R)`

- An interpretation function $[f(\mathbf{x})]_{R,e}$ from formulas in $\mathcal{F}(\Sigma, \mathbb{N}))$ in Coq statements

  `holds: (seq (carrier R))-> formula -> Prop`

- $R, e \models f$

  A proof of the Coq statement `(holds e f)`

- R is a model of the theory of (discrete) real closed fields

  $R : rcf$

Semantic quantifier elimination

A theory T on a language Σ with a set of variables $\mathcal{V}$ admits semantic quantifier elimination if

- for every $\phi \in \mathcal{F}(\Sigma, \mathcal{V})$,
- there exists a quantifier free formula $\psi \in \mathcal{F}(\Sigma, \mathcal{V})$
- such that for any model M of T, and for any list e of values,

$$M, e \models \phi \text{ iff } M, e \models \psi$$

This is the (a priori weaker) quantifier elimination result we formalize.

Semantic quantifier elimination

The formalization of the theorem in our framework is:

- Program a quantifier elimination procedure:

```
Fixpoint quantifier_elim :
  formula term -> formula term := ...
```

- Prove :

```
Lemma quantifier_elim_correct : forall (R_rcf : rcf),
  forall (f : formula term)(ctx : seq (R R_rcf)),
  (holds ctx f) <-> (holds ctx (quantifier_elim f)).
```

Decidability

There is a boolean test which given a context an a first-order formula, determines whether the formula holds or not:

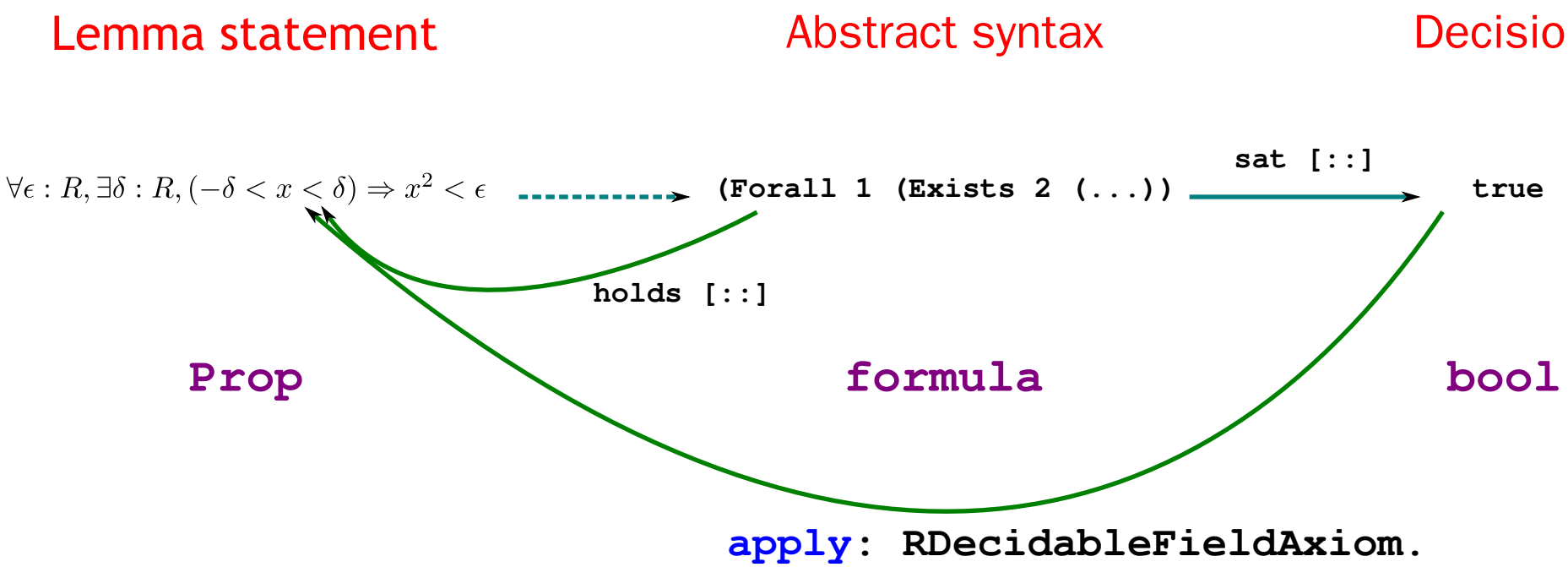
```
Definition DecidableField.axiom R
   (sat : seq (carrier R) -> formula -> bool) :=
   forall ctx f, (holds ctx f) <-> (sat ctx f = true).
```

If the semantic quantifier elimination is proved, program `sat` by:

- Eliminating quantifiers
- Deciding the obtained quantifier-free formula instanciated with parameters of the desired context

From proofs to proof-producing procedure

The (so-called) reflexion scheme:



Sufficient condition (for discrete structures)

Suppose it is possible to eliminate the $\exists$ in $\exists x, \bigwedge_{i=1}^{n} L_i$.

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  - The existential quantifier distributes over the disjunctions.

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  - F is equivalent to its disjunctive normal form: $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} L_{i,j}$
  - The existential quantifier distributes over the disjunctions.
  - The hypothesis is applied to every conjunction $\bigwedge_{j=1}^{m} L_{i,j}$

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    - ★ $\exists x, F$ where F can be considered qf (by induction).

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  - Existential case:
    - $\exists x, F$ where F can be considered qf (by induction).
    - The first lemma applies.

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- Now by induction on the structure of an arbitrary formula:
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  - Existential case: Ok

  - Universal case: $\forall x, F$ where F can be considered qf (by induction).
    - ★ Since $(F \vee \neg F)$ holds, $\forall x, F$ is equivalent to $\neg \exists x, \neg F$.
    - ★ $\neg F$ is quantifier free: the lemma applies to $\exists \neg F$.
    - ★ The outermost negation does not introduce quantifiers.

Sufficient condition (semantic version)

What we need to prove is that for any candidate theory R:

- There is a projection operator:

```
Parameter proj : nat -> formula -> formula.
```

- Such that:

```
Definition wf_proj_axiom proj := forall n f,
    literal_conjunction f -> qf_form (proj n f).

Definition holds_proj_axiom R proj :=
  forall n f ctx, literal_conjunction f ->
  (holds ctx (Exists n f)) <-> holds ctx (proj n f).
```

From language of fields to language of rings

Remark:

- Any first order sentence in this theory has an equivalent in the language of discrete ordered rings (possibly with more quantifiers, and we can allow rational constants).
- Hence:
  - For any first order sentence in the language of ordered rings
  - We want to construct a quantifier free formula in the language of rings which is equivalent to the former, in the theory of real closed fields.
  - We restrict our study to polynomial atoms.

Sufficient condition (geometric version)

Some vocabulary:

- Algebraic set: The set of roots of a finite number of polynomials.
  Realizations of an equality atom.
- Semi-algebraic set: The set of points satisfying a (disjunction of) finite number of polynomial inequalities (and equalities).
  Realization of quantifier-free formulas.
- Basic semi-algebraic set:

$$\{x \in R^k \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

Semi-algebraic sets are finite unions of basic semi-algebraic sets.

Sufficient condition (geometric version)

Theorem (Projection of semi-algebraic sets)

For any discrete real closed field R, for any S basic semi-algebraic set of R^{k+1} defined by polynomials with rational coefficients, the projection of S on R^k is a semi-algebraic set.

We only prove the theorem for basic semi-algebraic sets.

Emptiness of a one dimensional basic semi-algebraic set

Quantifier elimination amounts to deciding whether a basic semi-algebraic:

$$\{x \in R \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

is empty or not, for $P \in R[X]$ and $\mathcal{Q} \subset R[X]$ (finite).

The Tarski Query "oracle"

Definition (Tarski Query)

Let R be a real closed fields and $P, Q \in R[X]$ with $P \neq 0$. The Tarski Query of P and Q is defined as:

$$TQuery(P, Q) := \sum_{x \in R, P(x)=0} sign\ Q(x)$$

By sign we mean:

$$\begin{aligned} TQuery(P, Q) \quad := \quad & 1 \quad \times \sharp\{x \mid P(x) = 0 \wedge Q(x) > 0\} \\ + \quad & 0 \quad \times \sharp\{x \mid P(x) = 0 \wedge Q(x) = 0\} \\ + \quad & -1 \quad \times \sharp\{x \mid P(x) = 0 \wedge Q(x) < 0\} \end{aligned}$$

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Properties:

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In other words:

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Fortunately, $M := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ is invertible.

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$$\begin{bmatrix} \sharp\{P=0, Q=0\} \\ \sharp\{P=0, Q>0\} \\ \sharp\{P=0, Q<0\} \end{bmatrix} = M^{-1} \begin{bmatrix} TQuery(P,1) \\ TQuery(P,Q) \\ TQuery(P,Q^2) \end{bmatrix}$$

Tarski Queries and emptiness test

- The one dimensional basic semi-algebraic set

$$\{x \in R \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

is non empty iff $\sharp\{P = 0, Q > 0\} > 0$

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  is non empty iff $\sharp\{P = 0, Q > 0\} > 0$
- Tarski Queries provide an emptiness test for one sign condition.
- What about a basic semi-algebraic set with more sign conditions?

Multi Tarski Queries

Let $\mathcal{Q} := \{Q_1, \ldots, Q_n\} \subset R[X]$:

- A sign assignment for $\mathcal{Q}$ is:

$$\sigma : Q_i \mapsto \epsilon_i \in \{0, -1, 1\}$$

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- Let $\Sigma := \{0, -1, 1\}^{\mathcal{Q}}$ be the set of all possible sign assignments for $\mathcal{Q}$.

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- Let $\Sigma := \{0, -1, 1\}^{\mathcal{Q}}$ be the set of all possible sign assignments for $\mathcal{Q}$.
- Let $A := \{0, 1, 2\}^{\mathcal{Q}}$. For every $\alpha \in A$, let $\mathcal{Q}^{\alpha} := \prod Q_i^{\alpha(Q_i)}$

Multi Tarski Queries

Now define:

- $TQuery(P, \mathcal{Q}^A) := [TQuery(P, \mathcal{Q}^{\alpha_1}), \ldots, TQuery(P, \mathcal{Q}^{\alpha_{3^n}})]$ where the elements $\alpha_1, \ldots, \alpha_{3^N}$ of A are in lexicographic order.

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  where the elements $\alpha_1, \ldots, \alpha_{3^N}$ of A are in lexicographic order.
- $v(P = 0, \Sigma) := [\sharp\{P = 0, \sigma_1\}, \ldots, \sharp\{P = 0, \sigma_{3^n}\}]$
  where the elements $\sigma_1, \ldots, \sigma_{3^N}$ of Σ are in lexicographic order.

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- $(M_n)_{n \in \mathbb{N}}$ by $M_1 := M$ and $M_{k+1} := M_k \otimes M$

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Theorem

$$TQuery(P, \mathcal{Q}^A) = M_n.v(P = 0, \Sigma)$$

(and M_n is invertible)

Computing Tarski Queries

Theorem

For any two polynomials $P, Q \in R[X]$ with $P \neq 0$, the value of $TQuery(P, Q)$ only depends on the sign of the leading coefficients of polynomials occurring in the chain of remainders of P and $P'Q$ and on the degree of these remainders.

It is obtained as a the number of sign changes in the signed remainder chain.

Consequence on the emptiness test

- The emptiness of a basic semi-algebraic set

$$\{x \in R \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

  is determined by the degrees and signs of the leading coefficients of the polynomials occurring in the remainder chains of P and the $P'\mathcal{Q}^{\alpha}$.

- The emptiness of a basic semi-algebraic set

$$\{x \in R \mid \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

  is determined by the degrees and signs of the leading coefficients of the polynomials occurring in the remainder chains of C and C' and C and the $C''\mathcal{Q}^{\alpha}$, where $C = \prod_{i=1}^{n} Q_i$.

Back to quantifier elimination

Given a basic semi-algebraic set:

$$\{x \in R^{k+1} \mid P(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) > 0\}$$

we want to show that its projection on R^k is still a semi-algebraic set.

Back to quantifier elimination

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we want to show that its projection on R^k is still a semi-algebraic set. This problem is different from the unidimensional case:

$$\{(y, x) \in R^{k+1} \mid P(y)(x) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(y)(x) > 0\}$$

where $P, Q \in \mathbb{Q}[X_1, \ldots, X_k][X_{k+1}]$.

Back to quantifier elimination

Obtain a semi-algebraic description of

$$\{x \in R^k \mid \exists y \in R \mid P(x)(y) = 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x)(y) > 0\}$$

where $P, Q \in \mathbb{Q}[X_1, \ldots, X_k][X_{k+1}]$.

- Coefficients are no more in a field but in a ring. Yet euclidean division can be replaced by pseudo-euclidean division.

  Harmless

- The output semi-algebraic description depends on the values of the parameters.

  Example

Back to quantifier elimination

- We need a model-independent description of a finite partition of the space of parameters R^k into semi-algebraic cells.
- Each cell corresponds to a possible value for the quantifier free equivalent formula.
- This description is obtained by analyzing the tree of successive zero tests performed when computing the degrees and the pseudo divisions involved in the Tarski Queries.
- What about the (formal) formula construction?