Formalising the structure of extremal *p*-groups

Georges Gonthier

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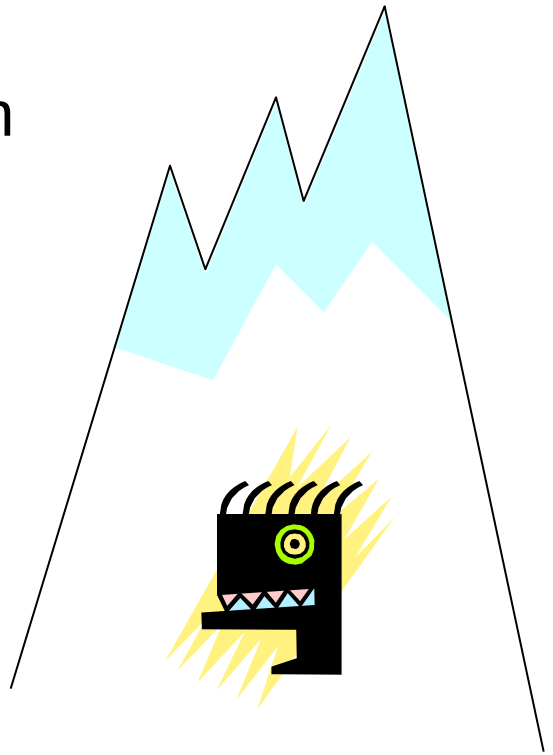
The Big Project

The Classification of Finite Simple Group

Frobenius groups
Thompson factorisation
character theory
linear representation
Galois theory
linear algebra
polynomials

Feit-Thompson

|G| odd
G simple
$G \approx F_p$



Goals and results

- Bring proof tools to mathematicians
- Apply software engineering to proof construction
- Prove landmark result
- Build on four-colour theorem experience

- Ssreflect proof language and Coq plugin
- Combinatorial components
- Linear algebra components
- Group theory components

4-color

Appel-Haken

finite groups

Feit-Thompson

Kepler

Hales

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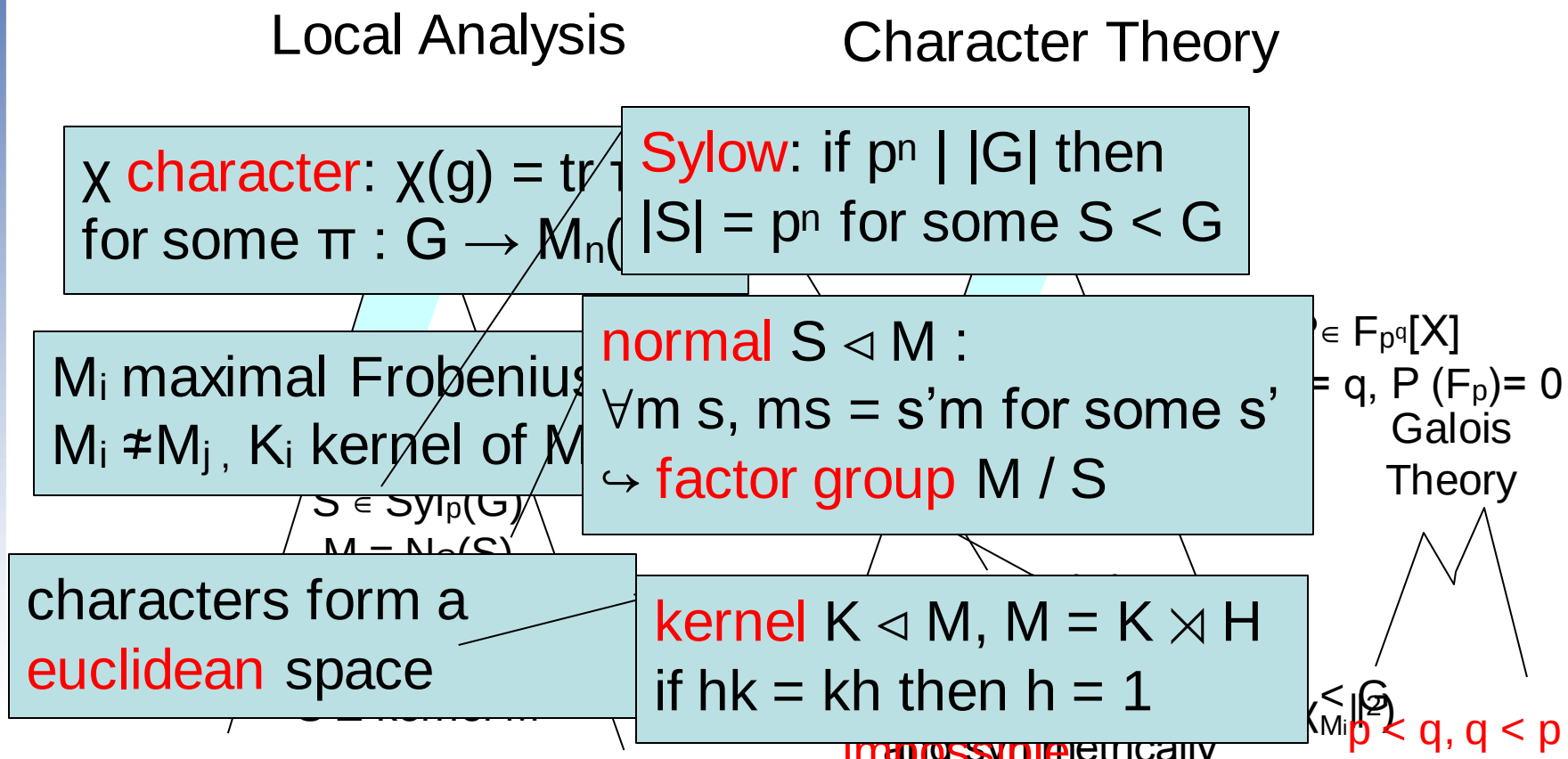
Tool review

- Data (inductive) types / propositions
- Computational reflection
  - compute values, types, and propositions
- Dependent types
  - first-class Structures
- Type / value inference
  - controlled by Coercion / Canonical Structure
- User notations

The Feit-Thompson proof

Let G be a minimal counter-example ...



Local Analysis

- Look at N(P), P a p-group, i.e. $P \rtimes N(P)$
- Use the nilpotent structure of P do prove stuff by induction on P and/or $A \subset N(P)$

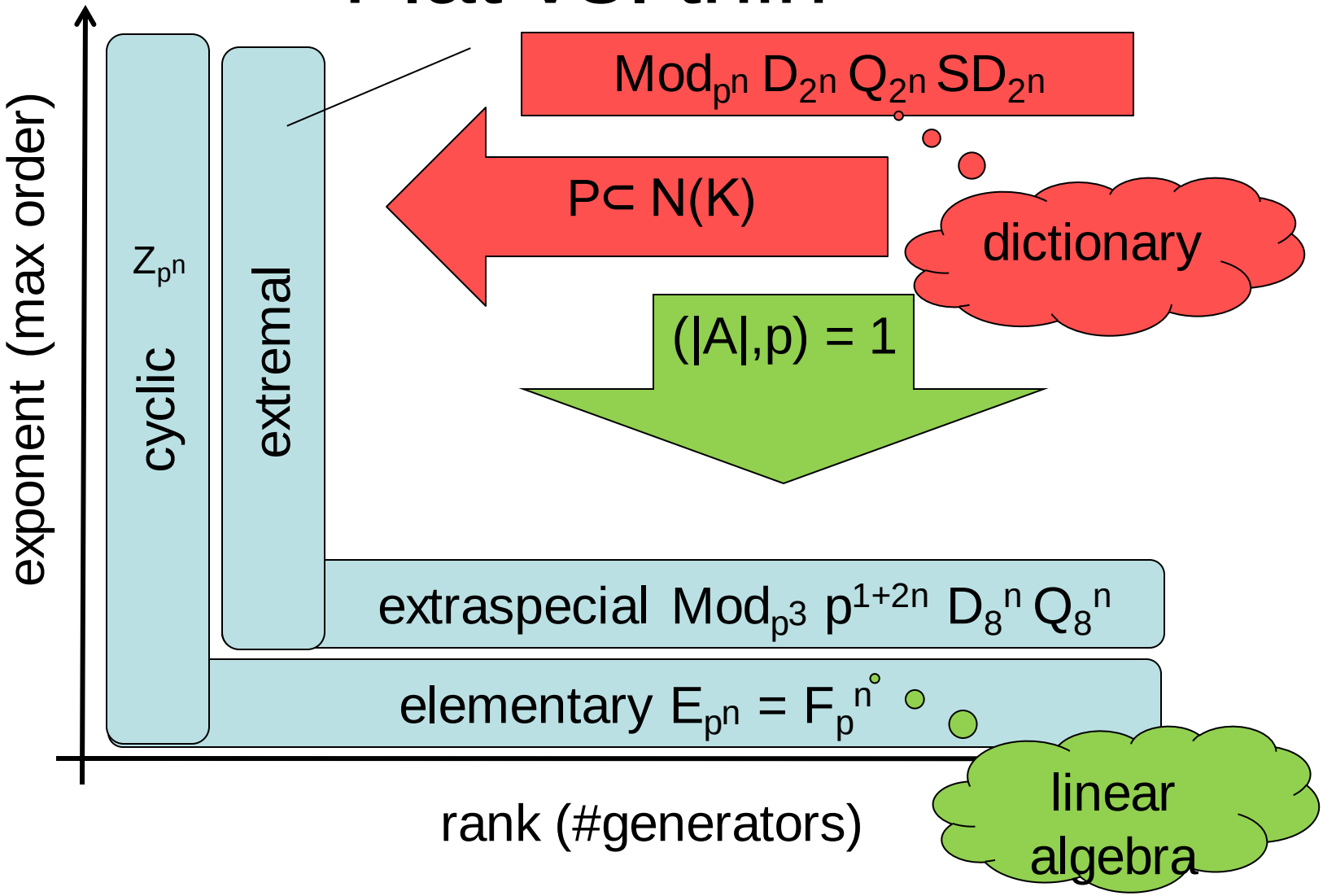
$$1 \lhd Z(P) \lhd Z_2(P) \lhd .... \lhd P$$

- Base cases: extraspecial and extremal p-groups.

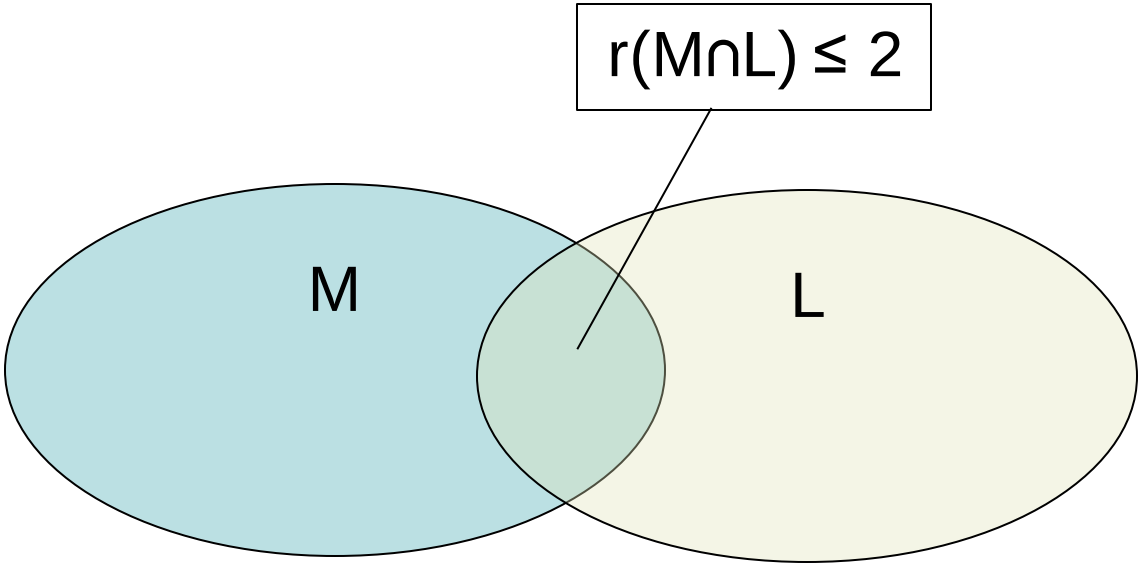
Roadmap

- Extremal p-groups
- Group presentations
- Big theorems

Flat vs. thin



The Uniqueness Theorem



Thompson transitivity:

$S \in Syl_p(G)$, $r(S) \geq 3$
maximal abelian $A \lhd S$
Q_1, Q_2 maximal in $\mathcal{U}(A, q)$
$Q_2 = Q_1^c$ for $c \in C_G(A)$

p-stability:

abelian $A \lhd S \in Syl_p(M)$
for some $C \lhd M$, $p \nmid |C|$
$A / C \lhd M / C$

Dictionary

```
Theorem dihedral2_structure :
  n > 1 -> extremal_generators G 2 n (x, y) -> G \isog 'D_m ->
[/\ [/\ X ><| Y = G, {in G :\: X, forall t, #[t] = 2}
      & {in X & G :\: X, forall z t, z ^ t = z^-1}],
    [/\ G ^`(1) = <[x ^+ 2]>, 'Phi(G) = G ^`(1), #|G^`(1)| = r
      & nil_class G = n.-1],
    'Ohm_1(G) = G /\ (forall k, k > 0 -> 'Mho^k(G) = <[x ^+ (2 ^ k)]>),
    [/\ yG :|: xyG = G :\: X, [disjoint yG & xyG]
      & forall M, maximal M G = pred3 X My Mxy M]
  & if n == 2 then (2.-abelem G : Prop) else
[/\ 'Z(G) = <[x ^+ r]>, #|'Z(G)| = 2,
     My \isog 'D_q, Mxy \isog 'D_q
   & forall U, cyclic U -> U \subset G -> #|G : U| = 2 -> U = X]].
```

Presentations

- $D_{2n} \cong$ Grp (x, y : $x^{2^{n-1}}$, y^2, x^y = x^-1)
- $SD_{8n} \cong$ Grp (x, y : x^{4n}, y^2, x^y = x^-1 * x^{2n-1})
- $Q_{2n} \cong$ Grp (x, y : x^{2n}, y^4, x^{2n-1} = y^2, x^y = x^-1)

... but this is not constructive (Post) !

```
Lemma Grp_dihedral :
 'D_m \isog Grp (x : y : (x ^+ q, y ^+ 2, x ^ y = x^-1)).
```

Faking free groups

- G ≈ Grp (x, y, ... : r, s, t)

  → (H ≲ G) ⇔ (H ≲ Grp (x, y, ... : r, s, t))
- H ≲ Grp (x, y, ... : r, s, t)

  → (H = ⟨x, y, ...⟩ & r = s = .. = 1)

```
Definition hom gT (B : {set gT}) p :=
  sat B 1 env1 (p (Cst 0)).
Definition iso gT (B : {set gT}) p :=
  forall rT (H : {group rT}), (H \homg B) = hom H p.
```

Faking relations

```
Inductive term :=
  | Cst of nat
  | Idx
  | Inv of term
  | Exp of term & nat
  | Mul of term & term
  | Conj of term & term
  | Comm of term & term.
```

```
Fixpoint eval {gT} e t : gT :=
 match t with
 | Cst i => nth 1 e i
 | Idx => 1
 | Inv t1 => (eval e t1)^-1
 | Exp t1 n => eval e t1 ^+ n
 | Mul t1 t2 => eval e t1 * eval e t2
 | Conj t1 t2 => eval e t1 ^ eval e t2
 | Comm t1 t2 => [~ eval e t1, eval e t2]
end.
```

```
Infix "*" := Mul : group_presentation.
Infix "^+" := Exp : group_presentation.
Infix "^" := Conj : group_presentation.
Notation "x ^-1" := (Inv x) : group_presentation.
```

Faking presentations

```
Inductive formula := Eq2 of term & term | And of formula & formula.
Inductive rel_type := NoRel | Rel vT of vT & vT.
Inductive type := Generator of term -> type | Formula of formula.
Inductive env gT := Env of {set gT} & seq gT.
```

```
Definition bool_of_rel r := if r is Rel vT v1 v2 then v1 == v2 else true.
Definition and_rel vT (v1 v2 : vT) r :=
  if r is Rel wT w1 w2 then Rel (v1, w1) (v2, w2) else Rel v1 v2.
 Fixpoint rel {gT} (e : seq gT) f r :=
  match f with
  | Eq2 s t => and_rel (eval e s) (eval e t) r
  | And f1 f2 => rel e f1 (rel e f2 r)
  end.
```

Using presentations

```
Lemma Grp'_dihedral :
  'D_m \isog Grp (x : y : (x ^+ 2, y ^+ 2, (x * y) ^+ q)).
Proof.
move=> gT G; rewrite Grp_dihedral.
apply/existsP/existsP=> [] [[x y]] /=.
   case/eqP=> <- xq1 y2 xy; exists (x * y, y).
   rewrite !xpair_eqE /= eqEsubset.
   ...
  by rewrite  y2 mulg1 xq1 !eqxx.
case/eqP=> <- x2 y2 xyq; exists (x * y, y).
....
by rewrite !mulgA mulgK -mulgA -!(expgS _ 1) x2 y2 mulg1.
Qed.
```

Proving presentations

```
Lemma isogEcard :
  forall rT aT (G : {group rT}) (H : {group aT}),
 (G \isog H) = (G \homg  H) && (#|H| <= #|G|).

Lemma card_dihedral : #|'D_m| = m.

Lemma normal_rank1_structure :
  forall gT p (G : {group gT}),
  p.-group G ->
  (forall X : {group gT}, X <| G -> abelian X -> cyclic X) ->
     cyclic G
 \/ [&& p == 2, extremal2 G & (#|G| >= 16) || (G \isog 'Q_8)].
```

Big Theorems

```
Theorem dihedral2_structure :
  n > 1 -> extremal_generators G 2 n (x, y) -> G \isog 'D_m ->
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   & forall U, cyclic U -> U \subset G -> #|G : U| = 2 -> U = X]].
```

Using big theorems

```
Lemma generators_2dihedral :
   n > 1 -> G \isog 'D_m ->
  exists2 xy, extremal_generators G 2 n xy
              & let: (x, y) := xy in #[y] = 2 /\ x ^ y = x^-1.


Lemma normal_rank1_structure :
   ....
have [[x y] genG _] := generators_2dihedral n_gt1 isoG.
have [_ _ _ [_ _ maxG]] := dihedral2_structure n_gt1 genG isoG.
```

Conclusions

- Formalizing big math sometimes requires big theorems.
- It is paramount to build usability into formal theories.
- Type theory and reflection are powerful and flexible tools engineer this usability.

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