Rings with Explicit Divisibility Formalized in Coq

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Goal

- Formalize constructive algebra in Coq
- Executable within Coq - can be used for computation in proofs

Overview

- Rings with explicit divisibility
  - GCD rings
  - Bezout rings
  - Euclidean rings
- Some polynomial theory
- Smith normal form
  - Constructive PIDs
- Standard examples: $\mathbb{Z}$ and $k[x]$ where k is a field

Formalization

- Formalized using the SSReflect extension to Coq
- Based on chapter 4, Divisibility in discrete domains, in *A course in constructive algebra* by Mines, Richman and Ruitenburg

Rings with explicit divisibility

- A ring R has *explicit divisibility* if it has a divisibility test that give witnesses:

$$a \mid b \leftrightarrow \exists x. b = xa$$

DvdRing

```
CoInductive div_spec (a b : R) : option R -> Type :=
| Dvd x of a = x * b : div_spec a b (Some x)
| NDvd of (forall x, a != x * b) : div_spec a b None.

Record mixin_of (R : ringType) : Type := Mixin {
  div : R -> R -> option R;
  _ : forall a b, div_spec a b (div a b)
}.
```

GCD rings

- A ring R is a *GCD ring* if every pair of elements have a greatest common divisor

$$\forall ab.\exists g.(g \mid a) \land (g \mid b) \land (\forall g'.g' \mid a \land g' \mid b \rightarrow g' \mid g)$$

GcdRing

```
Record mixin_of (R : dvdRingType) : Type := Mixin {
  gcd : R -> R -> R;
  _ : forall a b g,
    g %| gcd a b = (g %| a) && (g %| b)
}.
```

Bezout rings

- Non-Noetherian analogue of principal ideal domains
- Principal ideal domains: Every ideal is principal
- Bezout ring: Every *finitely generated* ideal is principal
- Equivalent definition:

$$\forall ab.\exists xy.ax + by = gcd(a, b)$$

BezoutRing

```
CoInductive bezout_spec (a b : R) : R * R -> Type :=
  BezoutSpec x y of
   gcdr a b %= x * a + y * b : bezout_spec a b (x, y)

Record mixin_of (R : gcdRingType) : Type := Mixin {
  bezout : R -> R -> (R * R);
   _ : forall a b, bezout_spec a b (bezout a b)
}.
```

Euclidean rings

- Euclidean norm: $f : R \to \mathbb{N}$
- Euclidean division: $\forall ab.\exists qr.a = bq + r$ and either $f(r) < f(b)$ or $r = 0$.
- Examples: $\mathbb{Z}$ with absolute value and $k[x]$ with degree

Some polynomial theory

- If R has explicit divisibility then $R[x]$ also has
- If R is a GCD ring then $R[x]$ also is a GCD ring
- This proof is based on the presentation in *The Art of Computer Programming* by Knuth and **it doesn't use the field of fractions** of R as in Mines, Richman, Ruitenburg
- Give GCD algorithm for $\mathbb{Z}[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_n]$

Smith normal form

- Generalization of Gauss elimination algorithm to work over any principal ideal domain instead of field
- Given a matrix M compute invertible matrices L and R such that LMR is diagonal and $a_{ii} \mid a_{(i+1)(i+1)}$
- Motivation: Computation of homology groups of simplicial complexes from algebraic topology

Constructive PIDs

- In order to formalize the Smith normal form algorithm we need constructive approximation of principal ideal domains
- Mines, Richman, Ruitenburg: A *constructive PID* is a Bezout domain such that if we have a sequence of $u(n)$:s with $u(n+1) \mid u(n)$ then there is some k such that $u(k) \mid u(k+1)$
- Formalized in type theory by having that strict divisibility is well founded

Constructive PIDs

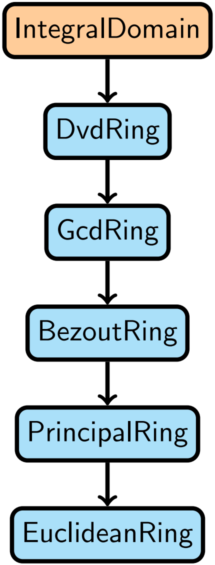
```
Definition sdvdr (R : dvdRingType) (x y : R) :=
  (x %| y) && ~~(y %| x).

Record mixin_of (R : dvdRingType) : Type := Mixin {
  _ : well_founded (@sdvdr R)
}.
```

Constructive PIDs

- This has been used to implement GCD algorithm showing that constructive PIDs are GCD rings
- Vincent Siles used this to implement Smith normal form algorithm in SSReflect

Overview


IntegralDomain
DvdRing
GcdRing
BezoutRing
PrincipalRing
EuclideanRing

Examples

- Have proved that $\mathbb{Z}$ and $k[x]$ where k is a field are Euclidean rings and hence the other structures as well

```
> Time Eval compute in (gcdr 11466 1428)%Z.
     = 42%Z
Finished transaction in 0. secs (0.109993u,0.s)

> Time Eval compute in (123123 %/ 1234)%Z.
     = 99%Z
Finished transaction in 0. secs (0.013333u,0.s)
```

Future work

- Efficient implementation of polynomials
- Implement executable version of Smith normal form algorithm
- Certified computation of homology groups of simplicial complexes

Questions?

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