# What is $\pi$?

Yves Bertot

August 2012

# Introduction

- A history of $\pi$
- Modifying the library: removing axioms
- Computing $\pi$

# Historically

- $\pi$ is the surface of the circle, or the perimeter
- How does one model surface computation?
- Main tool: computing the surface under a curve
  - Riemann or Newton integration
- Curve : $y = \sqrt{(}1 + x^2)$

# Coq standard library's approach

- Calculus: limits, derivatives, power series
- Alternated series
- Sine and Cosine functions defined as power series

# Sine and Cosine definitions

- $\cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}$
- $\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$
- In formalized mathematics, we have to show that the limits exists

# Formal definitions

```
Definition infinite_sum (s:nat -> R) (l:R) : Prop :=
  forall eps:R, eps > 0 ->
    exists N : nat, forall n:nat, (n >= N)%nat ->
     R_dist (sum_f_R0 s n) l < eps.

Definition cos_n (n:nat) : R :=
  (-1) ^ n / INR (fact (2 * n)).

Definition cos_in (x l:R) : Prop :=
  infinite_sum (fun i:nat => cos_n i * x ^ i) l.

Lemma exist_cos : forall x:R, { l:R | cos_in x l }.
(* Proof using a general lemma: d'Alembert's theorem *)

Definition cos (x:R) : R :=
  let (a,_) := exist_cos (Rsqr x) in a.
```

# Where is the circle?

- Lemma `cos_plus` : $\forall x, \cos(x+y) = \cos x \cos y - \sin x \sin y$
  - Long and difficult proof, ad hoc
  - Should have done a general lemma (Mertens for the product of series)
- Lemma `sin2_cos2` : $\forall x, \sin^2 x + \cos^2 x = 1$
  - Easy consequence
- Down to here, everthing is nice, but where is $\pi$

# Standard library definition of $\pi$ until 2012

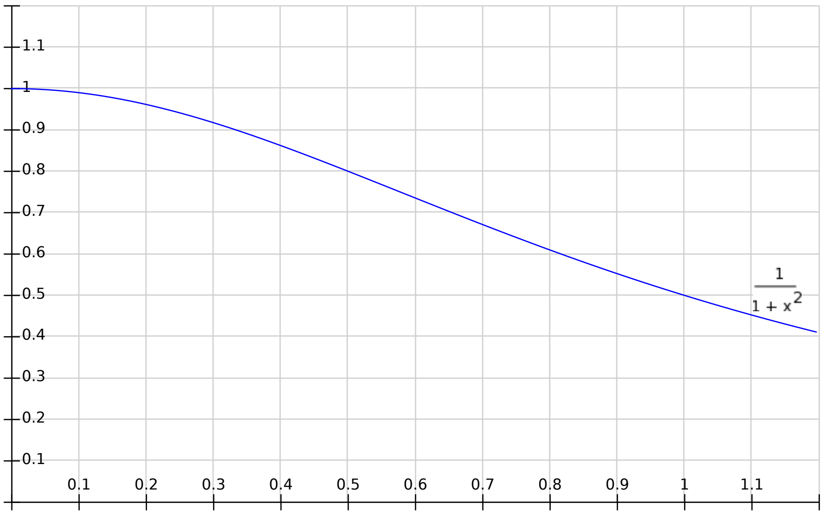
- $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1}$
- An alternated series: the proof of convergence is easy
- Converges slowly, but good enough to show that $3 < \pi < 4$
  - $\pi < 4$ at the first term, $3 < \pi$ at the $8^{th}$
- Why choose this definition?

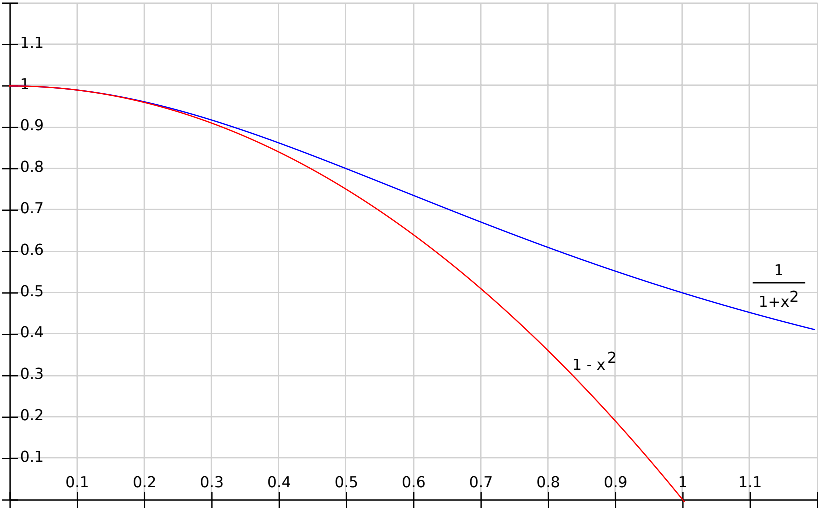
# Required properties of $\pi$ for trigonometry

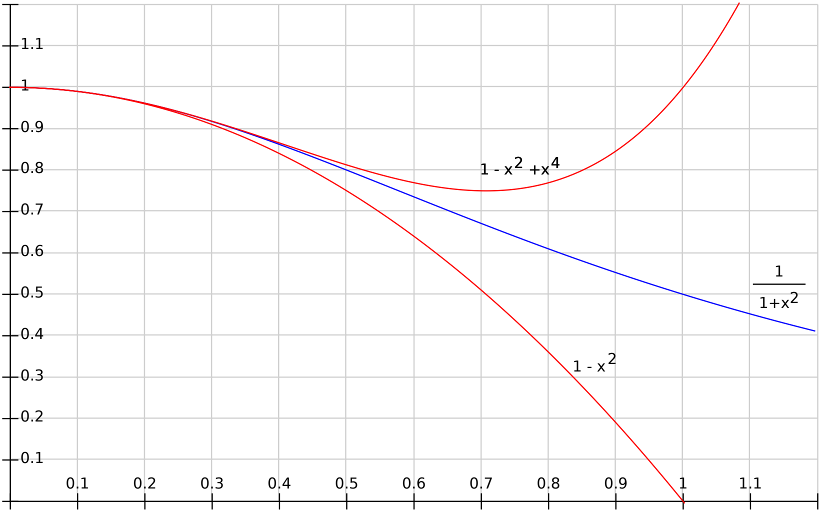
- $\sin \frac{\pi}{2} = 1$ : admitted as an axiom
- $\cos \frac{\pi}{2} = 0$: Important property
- $\cos(\frac{\pi}{2} - x) = \sin x$
- Deduce all properties of sin from properties of cos
- Proof of the axiom: work done by G. Allais

# Proving the axiom : The tangent function

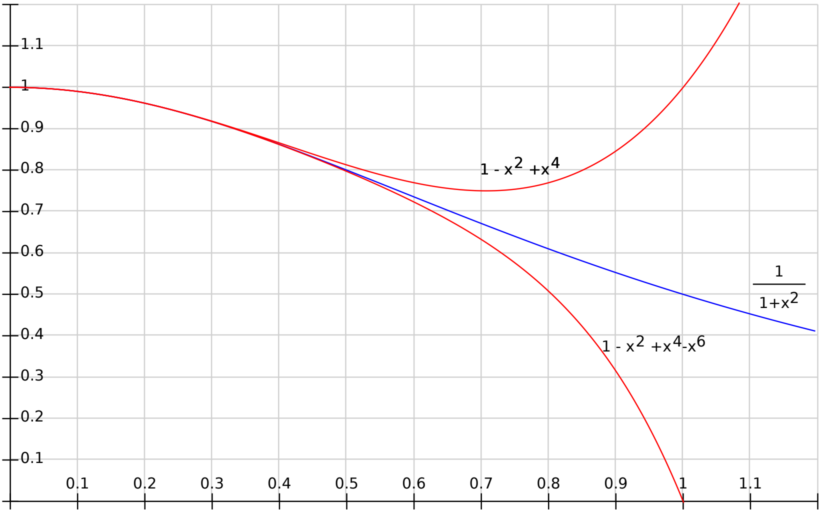
- Important properties of tan
- $\tan(\frac{\pi}{4}) = 1$, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$, $\tan' x = 1 + \tan^2 x$
- Move interest to the inverse function of tan, called *atan*.
- $atan' \ x = \frac{1}{1+x^2}$
- Taylor expansion : $\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots (-1)^i x^{2i}$
- Integrating gives : $atan \ x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots (-1)^i \frac{x^{2i+1}}{2i+1}$
- We want to compute *atan* 1


1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
$\frac{1}{1+x^2}$

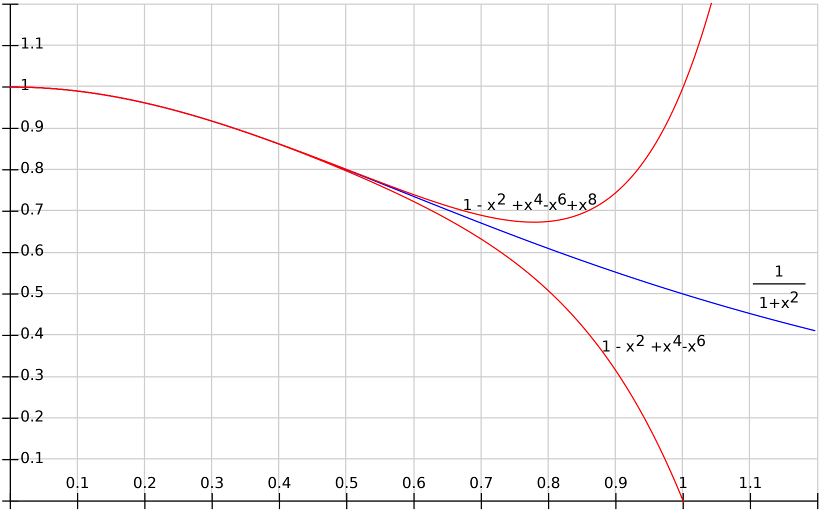

1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1
$\frac{1}{1+x^2}$
$1 - x^2$



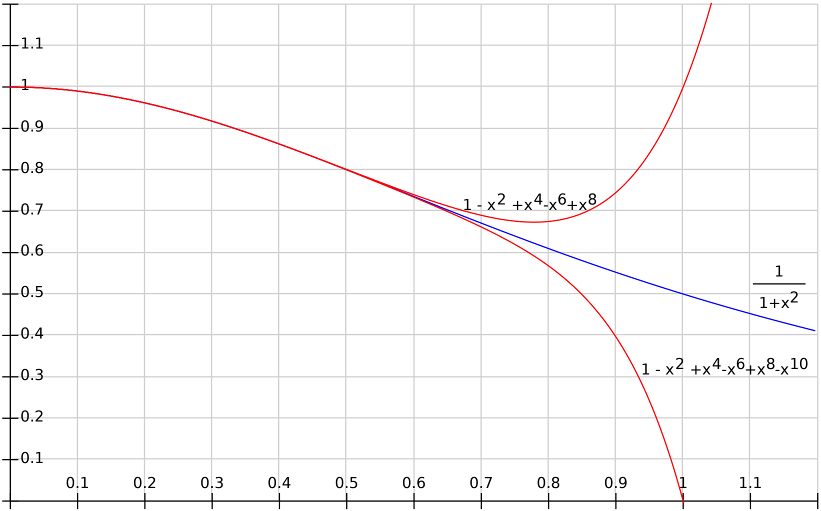
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1 - x2 +x4
1
1+x2
1 - x2



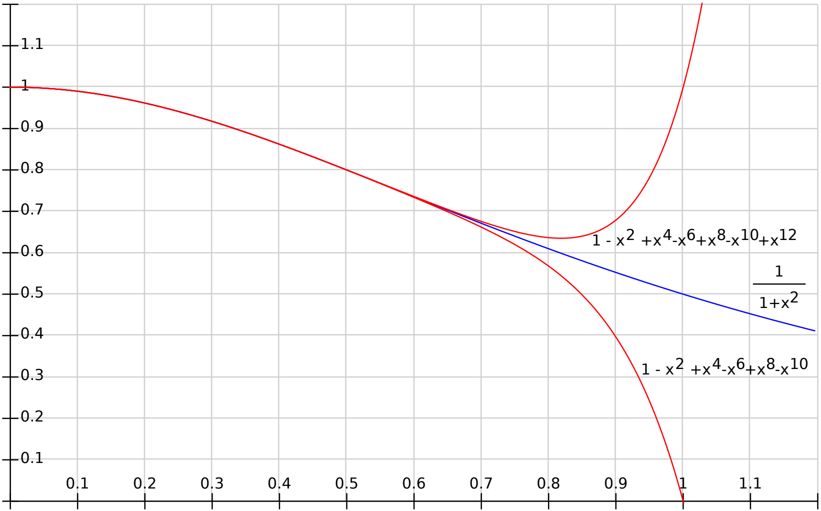
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1
$1 - x^2 + x^4$
$\frac{1}{1+x^2}$
$1 - x^2 + x^4 - x^6$



1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1
$1 - x^2 + x^4 - x^6 + x^8$
$\frac{1}{1+x^2}$
$1 - x^2 + x^4 - x^6$



1 - x2 +x4-x6+x8
1
1+x2
1 - x2 +x4-x6+x8-x10
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1



1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1 - x2 +x4-x6+x8-x10+x12
1 / 1+x2
1 - x2 +x4-x6+x8-x10

# Removing the axiom an elusive target

- Nice proof by G. Allais
- Reciprocal functions and their derivatives
- The power series $\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{2i+1}$ converges between 0 and 1
- The power series and *atan* coincide in this interval
- But all properties of sin, hence tan and *atan* depend on the axiom

# How to write the sums

```
Lemma lower_bound_atan :
  forall x n, 0 < x < 1 ->
    sum_f_R0 (fun i => (-1)^ i * x^(2 * i + 1) /
       INR (2 * i + 1)) (2 * n + 1)
   < atan x.
```

- Difficulty with many types of numbers
- This proof done by comparing the derivatives
- Proofs about derivability often clumsy

# Removing the axiom : drastic step

- Choose a new definition for $\pi$
- Show that cos has a single root $p$ between $0, 2$
  - Between 0 and 2, $\sum_{i=0}^{2*k+1} \frac{(-1)^i x^{2i}}{(2i)!} < \cos x < \sum_{i=0}^{2*k+1} \frac{(-1)^i x^{2i}}{(2i)!}$
  - Thus $\cos 2 < 0$
  - We also have $0 < \cos 1$, and cos is continuous
  - The Intermediate value theorem gives a root between 1 and 2
- Define $\pi$ as $2p$
- All properties of sin remain, but no axiom anymore
- Proofs about tan, *atan* go through and justify the power series

# Machin formulas

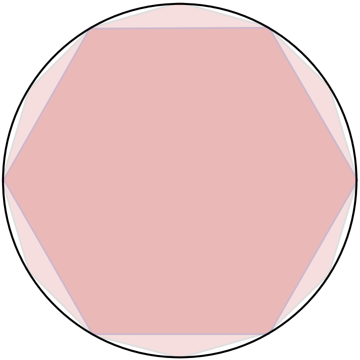
- Better to compute *atan* for numbers smaller than 1
- Use the second trigonometric property of *tan*
- $atan\frac{1}{5} + atan\frac{2}{3} = atan\ 1 = \frac{\pi}{4}$
- $atan\frac{1}{5} + atan\frac{7}{17} = atan\frac{2}{3} \ldots$
- After more computations : $\frac{\pi}{4} = 4atan\frac{1}{5} - atan\frac{1}{239}$
- This was used in 1706 to compute the first 100 decimals of $\pi$
- Computes fairly fast in Coq
- Other Machin formulas can be obtained
  - $atan\frac{1}{2} + atan\frac{1}{3}$ $\quad 2atan\frac{1}{3} + atan\frac{1}{7}$
    $3atan\frac{1}{4} + atan\frac{1}{20} + atan\frac{1}{1985}$
    $44atan\frac{1}{57} + 7atan\frac{1}{239} - 12atan\frac{1}{682} + 24atan\frac{1}{12943}$
    (last one taken from Krebbers and Spitters)

# What about the area of the circle?

- Area under the curve $y = \sqrt{1 - x^2}$ between -1 and 1
- Necessary link to *asin*, easily formalized by mirroring *atan*
- Small difficulty: *asin* is defined only between $-1$ and 1, *atan* is defined over all $\mathbb{R}$
- Propose to re-design IVT to accept functions only locally continuous and derivable

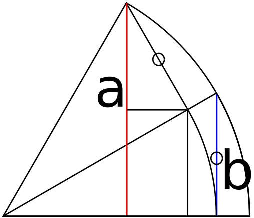
```
Lemma derivable_pt_lim_asin :
  forall x, -1 < x < 1 ->
  derivable_pt_lim asin x (1/sqrt (1 - x^2)).
```

# Approximation by inscribed polygons



# Archimedes' approximations by polygons

- I don't know what Archimedes did, but...
- You can compute the surface of triangles as used in the polygons
- $a = 2b\sqrt{1-b^2}$; $b = \sqrt{\frac{1-\sqrt{1-a^2}}{2}}$
  - Similar formula for a polygon outside



# Proving and computing Archimedes' sequence

- Starting with an hexagon $a_0 = \frac{\sqrt{3}}{2}$, $a_1 = \frac{1}{2}$
- Successive values : 2.6, 3, 3.1, 3.132, 3.139, 3.141
- Proof of correctness: we show that the sequence is $3 \times 2^n \sin \frac{\pi}{3 \times 2^n}$
- This sequence can be computed in Coq, using `Zsqrt` to approximate $\sqrt{\ }$ (not proved correct)

# Coq code for Archimedes's sequence

```
Definition Qsqrt x p : Q :=
Zsqrt_plain ((Qnum x * 10 ^ (2 * p))/(QDen x)) # 10 ^ p.

Fixpoint Qarch n p : Q :=
match n with
 0%nat => Qsqrt (3#4) p
|S m => Qsqrt ((1 - Qsqrt(1 - Qarch m p^2) p) * (1#2)) p
end.

Definition compute_pi_archimedes n p:=
  dec (((3 * 2 ^ n)#1) * Qarch n p) (Z_of_nat p).
```