Formalized foundations of polynomial real analysis

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Purpose of this formalization

Formalize real polynomial analysis **in the** SSReflect **extension of the** Coq **proof assistant**.

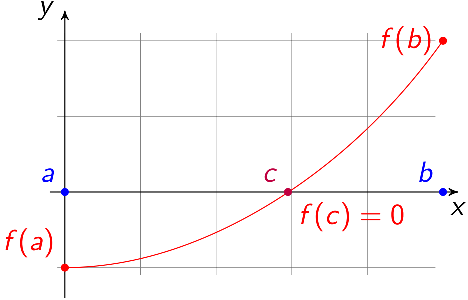
SSReflect provides a lot of **tools** and uses a lot of specific **programming techniques** in the domain of *finite groups* and *combinatorics*.

Reuse theses techniques to handle more « continuous » theories.

Real Closed Fields

Algebraic structure of reals : *Real Closed Fields* (*RCF*)

Field + Ordered + Intermediate value theorem for polynomials



Decidable Equality

In SSReflect, structures have decidable equality.
We can define this (implicit) coercion in Coq

```
Coercion is_true (b : bool) : Prop := (b = true).
```

SSReflect

- uses intensively this coercion
- has facilities to go from one point of view to the other (bool-Prop reflection).

We then see boolean equality as propositional equality, for free.

What do we need ?

- ⇒ Make case analysis on $x \leq y$
- ⇒ Combine statements (using transitivity with both $\leq$ $<$, compatibility with operations, etc ..)
- ⇒ Speak about signs and absolute value
- ⇒ Use max and min

Taking advantage of the boolean predicate

Making `le` a boolean predicate.

Like before, consider this boolean predicate as proposition through the coercion `is_true`

⇒ Use equalities to rewrite expressions with order

- `(x+z <= y+z)= (x <= y)`
- `(sign x == 1)= (0 < x)`
- ...

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- `(x+z <= y+z)= (x <= y)`
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- ...

⇒ Use `if x <= y then ... else ...` in programs

Tools

- Multiple lemmas about transitivity and compatibility between `le`, `lt` and field operations

$\Rightarrow$ Need for good naming conventions.

Strict comparison

We'll define the strict order `lt` from the large one `le` by :

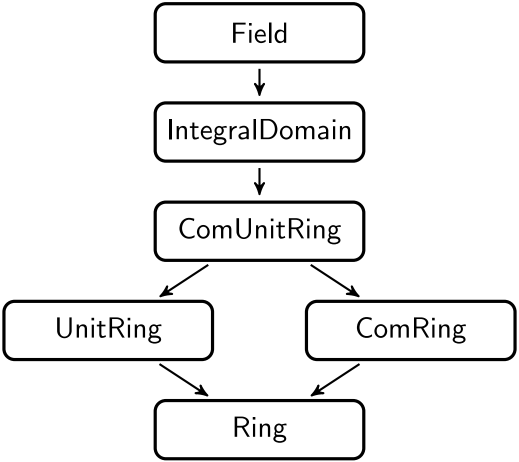
```
Definition lt x y := ~~ (le y x).
```

and prove its properties.

Ordered ring mixin

```
Record mixin_of (R : ringType) := Mixin {
  le : rel R;
  _ : antisymmetric le;
  _ : transitive le;
  _ : total le;
  _ : forall z x y, le x y -> le (x + z) (y + z);
  _ : forall x y, le 0 x -> le 0 y -> le 0 (x * y)
}.
```

Integration in existing SSReflect algebraic hierarchy



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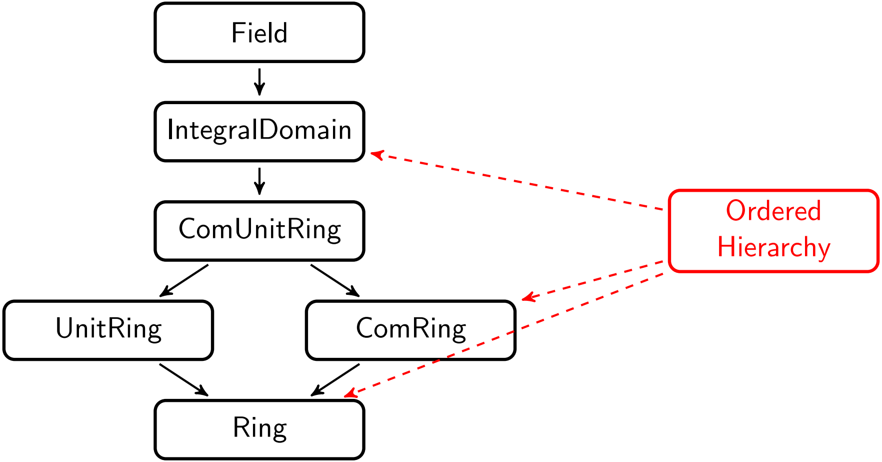
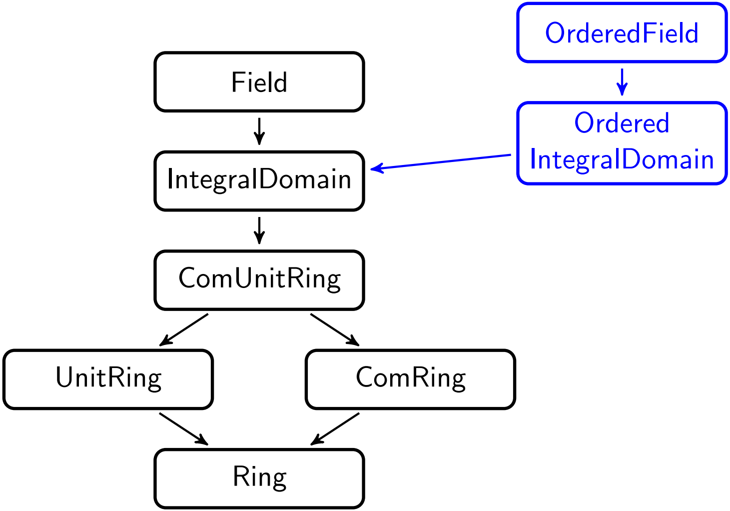


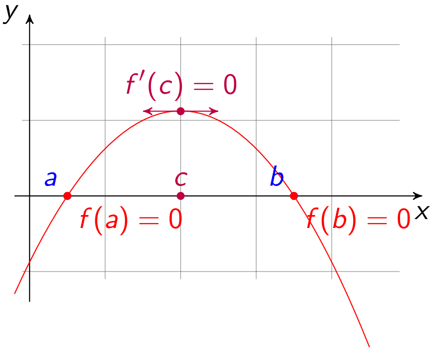
Integration in existing SSReflect algebraic hierarchy



Rolle Theorem for polynomials

A first hint that *RCF* is a good abstraction of reals :
We are able to prove :

```
Lemma rolle : forall (a b : R) (p : {poly R}),
  a < b -> p.[a] = 0 -> p.[b] = 0 ->
  exists c, a < c < b /\ p^`().[c] == 0.
```



Sketch of the constructive proof

```
Lemma rolle_weak : forall (a b : R) (p : {poly R}),
   a < b -> p.[a] = 0 -> p.[b] = 0 ->
   exists c , a < c < b
        /\ (p^`().[c] = 0 \/ p.[c] = 0).
```

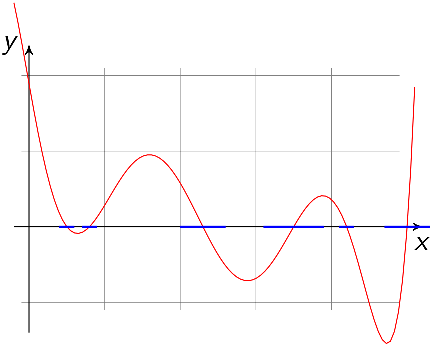
And conclude `rolle` from it by iterating `rolle_weak`. It terminates because P has less than $\deg(P)$ roots.

What else can we do with *IVT*?

Particularly useful examples

- Rolle Theorem
- Mean Value Theorem
- Write a function that computes the real roots of any polynomial
- Prove that given a polynomial P, and a root x of P, one can find a neighborhood of x on which P has no root except x.
- ...

Isolation of roots



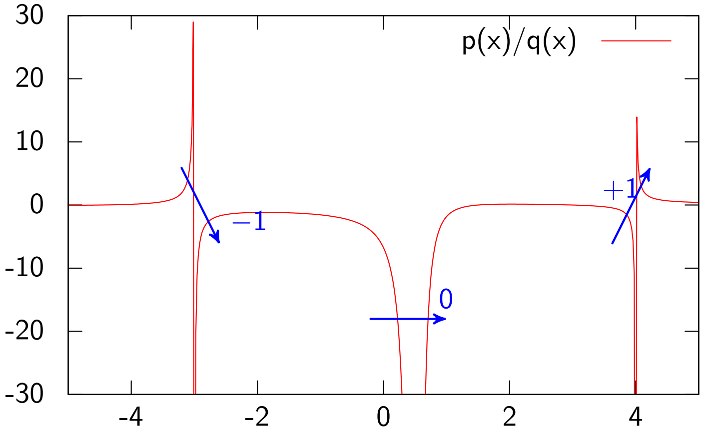
Towards quantifier elimination

First step of Quantifier Elimination in *RCF*.
Which entails decidability of the theory of *RCF*.

Let's pick one concept from it : Cauchy Index (proof almost done).

$$\mathrm{CInd}(\frac{P}{Q},]a, b[) =$$

$$\text{number of positive jumps} - \text{number of negative jumps}$$



Useful property of Cauchy Index

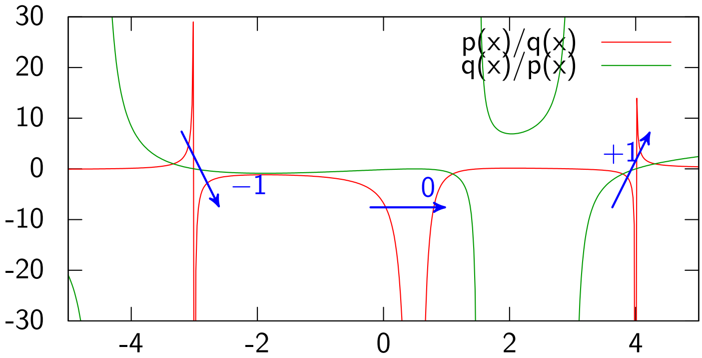
Property

If $P(a), P(b), Q(a), Q(b) \neq 0$ then,

$$\mathrm{CInd}\left(\frac{P}{Q},]a,b[\right) + \mathrm{CInd}\left(\frac{Q}{P},]a,b[\right) =$$

$$\begin{cases} \mathrm{sign}\,(PQ(b)) & \text{if } PQ(a)PQ(b) < 0 \\ 0 & \text{else} \end{cases}$$

Idea of the proof : combinatorics



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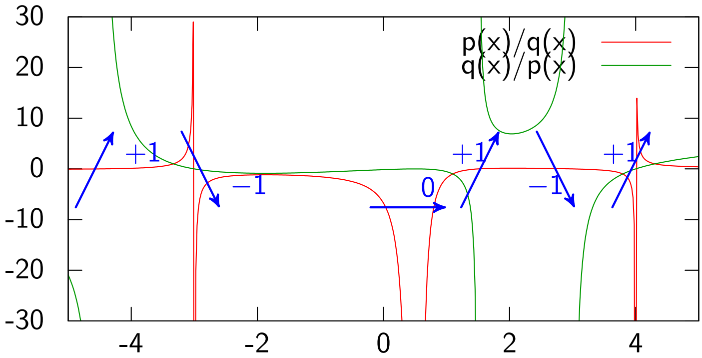

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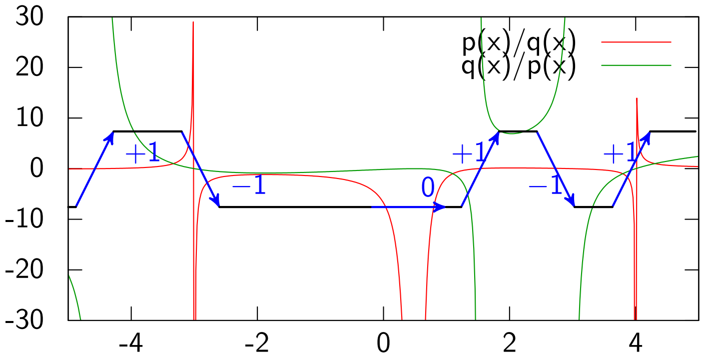

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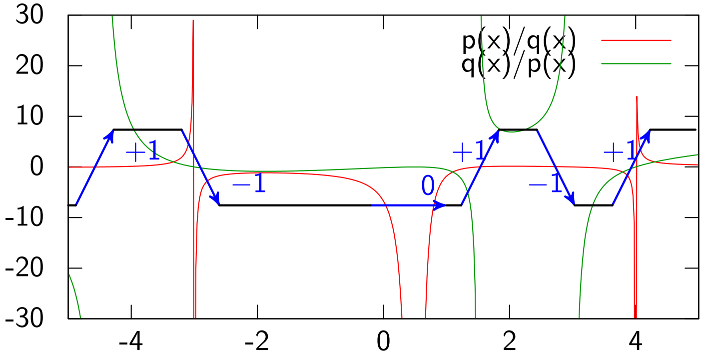
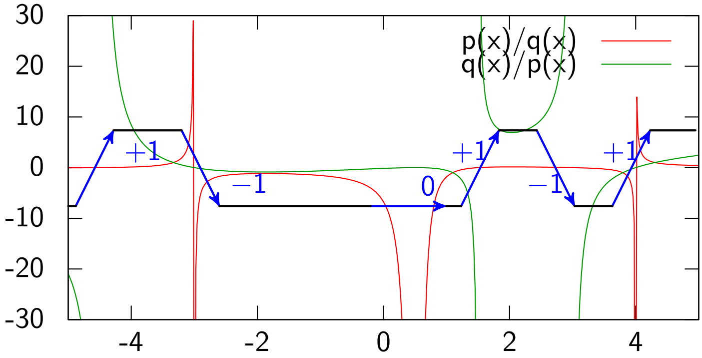


Jumps in the list of *signs* of PQ. `[-1;1;-1;-1;1;-1;1]`

Trick of the proof

The sum of jumps of a list $l = x_0, \ldots, x_n \in \{-1, 1\}^*$ verifies a useful property : it's the jump between x_0 and x_n.
i.e.

$$\begin{cases} \mathrm{sign}(x_n) & \text{if } x_0 x_n < 0 \\ 0 & \text{else} \end{cases}$$



Jumps in the list of *signs* of PQ. [-1;1;-1;-1;1;-1;1]
Jump between the first sign −1 and the last one 1, i.e.

$$\begin{cases} \text{sign}\,(PQ(b)) & \text{if } PQ(a)PQ(b) < 0 \\ 0 & \text{else} \end{cases}$$

Conclusion

A library which provides usable tools.
It is used in works in progress on

- Quantifier elimination in *RCF*
- Formalisation of Bernstein Polynomials

What next ?

- Instantiate the *Real Closed Fields* Structure
- Prove some reflexive tactics using it
- ... to provide a little more automation
- Generalize notion of continuity in this context
- Extend to further real analysis

The End

Thank you for your attention. Any questions ?