Experiments with computable matrices in the Coq system[1]

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SSReflect matrices

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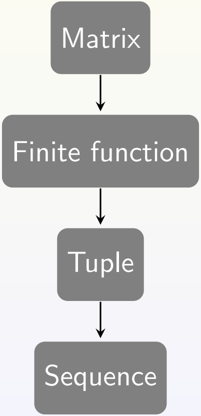

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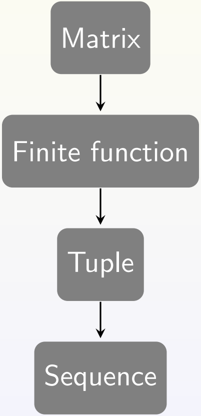

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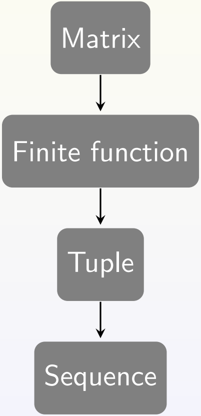

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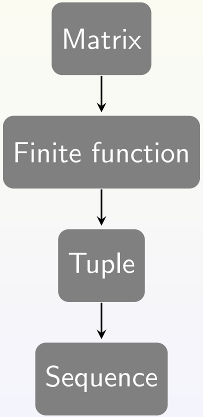

- Fine-grained architecture
- Easier to get the properties
- Had to be locked to avoid term size explosion

Our objective

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Parametrized datatypes are useful in mathematical description, they can reduce the need for side conditions in specifications.

```
Variable M : matrix R (m1 + m2) (n1 + n2).
Lemma submxK :
block_mx (ulsubmx M) (ursubmx M) (dlsubmx M) (drsubmx M) = M.
```

A traditional approach to vectors

```
Inductive vect (A : Type) : nat -> Type :=
| Vnil : vect A 0
| Vcons : forall n, A -> vect A n -> vect A (S n).
```

- Traditionnally used to justify dependent datatypes
- Refers to a known weakness of conventional programming languages
  - Array overflow
  - Static verification of bound checking

```
Vnth : forall n p, p < n -> vect A n -> A
```

Problems with binary functions

```
Fixpoint Vzip A B C (op : A -> B -> C) n (v : vect A n) :=
  match v in vect _ n return vect B n -> vect C n with
    Vnil => fun w => Vnil
  | Vcons n' x v' => fun w : vect B (S n') =>
    match w in vect _ k return k = S n' -> vect C k with
      Vnil => fun _ => Vnil
    | Vcons p' y w' => fun h =>
      Vcons (op x y)
         (Vzip A B C op p'
            (eq_rect n' (vect A) v' _ (eqS h)) w')
    end (refl_equal _)
  end.
```

Problems with casts

- the expression `(eq_rect n' (vect A) v' p' ...)` changes the type of `v'`, not the value,
- It is a formally verified type cast,
- This clutters statements,
- Human beings prefer to be typeless in these conditions
- Proofs also become unwieldy
  - especially because of (refl_equal _)

Relaxed implementation

- Use directly unsized lists (or sequences in `ssreflect` jargon)
- No size guarantee expressed by result type
- The procedure terminates when the end of the shorter list is reached

```
Fixpoint zip A B C (op : A -> B -> C)
   (l1 : list A) (l2 : list B) :=
match l1, l2 with
  a::l1', b::l2' => op a b::zip A B C op l1' l2'
| _, _ => nil
end.
```

A library of effective matrices

Abstract definitions

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Abstract definitions

Algorithmic refinement

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Abstract definitions

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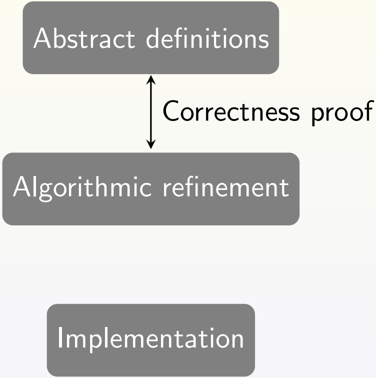
Implementation

A library of effective matrices



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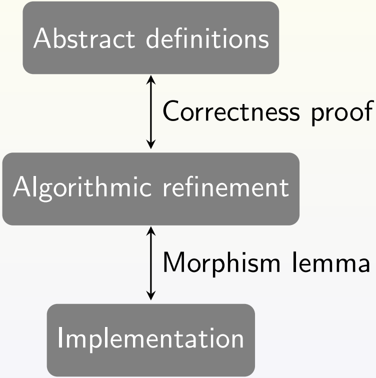

List-based representation of matrices

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Variable R : ringType.
Definition seqmatrix := seq (seq R).
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```

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Lemma seqmx_of_funE : forall (f : 'I_m -> 'I_n -> R),
  seqmx_of_mx (\matrix_(i < m, j < n) f i j) = mkseqmx_ord f.
```

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Lemma seqmx_eqP: forall (M N:'M[mT]_(m,n)),
  seqmx_of_mx M == seqmx_of_mx N -> M = N.
```

Large scale reflection on matrices

```
Goal \matrix_(i, j < n) (f i j) = \matrix_(i, j < n) (h i j).
Proof.
by apply/seqmx_eqP ; rewrite seqmx_of_funE ; native_compute.
Qed.
```

Addition

```
Definition addseqmx (M N : seqmatrix) : seqmatrix :=
  map2 (map2 (fun x y => x + y)) M N.

Lemma addseqmxE:
  {morph (@seqmx_of_mx m n) : M N / M + N >-> addseqmx M N}.
```

Naive product

```
Definition mulseqmx (M N: seqmatrix) : seqmatrix :=
  let N := transposeseqmx N in
  let f := foldl2 (fun z x y => x * y + z) 0 in
  map (fun r => map (f r) N) M.

Lemma mulseqmxE (M:'M_(m,p)) (N:'M_(p,n)) :
  mulseqmx (seqmx_of_mx M) (seqmx_of_mx N) =
    seqmx_of_mx (M *m N).
```

Winograd (1971)

$$\left(\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right) \times \left(\begin{array}{c|c} B_{1,1} & B_{1,2} \\ \hline B_{2,1} & B_{2,2} \end{array}\right) = \left(\begin{array}{c|c} C_{1,1} & C_{1,2} \\ \hline C_{2,1} & C_{2,2} \end{array}\right)$$

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$$\begin{array}{lll}
S_1 = A_{2,1} + A_{2,2} & P_1 = A_{1,1} \times B_{1,1} & U_1 = P_1 + P_6 \\
S_2 = S_1 - A_{1,1} & P_2 = A_{1,2} \times B_{2,1} & U_2 = U_1 + P_7 \\
S_3 = A_{1,1} - A_{2,1} & P_3 = S_4 \times B_{2,2} & U_3 = U_1 + P5 \\
S_4 = A_{1,2} - S_2 & P_4 = A_{2,2} \times T_4 & C_{1,1} = P_1 + P_2 \\
T_1 = B_{1,2} - B_{1,1} & P_5 = S_1 \times T_1 & C_{1,2} = U_3 + P_3 \\
T_2 = B_{2,2} - T_1 & P_6 = S_2 \times T_2 & C_{2,1} = U_2 - P_4 \\
T_3 = B_{2,2} - B_{1,2} & P_7 = S_3 \times T_3 & C_{2,2} = U_2 + P_5 \\
T_4 = T_2 - B_{2,1} & &
\end{array}$$

Fast matrix product

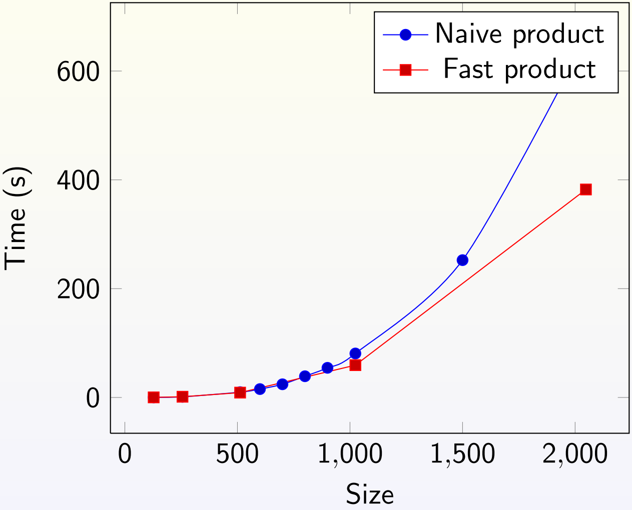
```
Fixpoint winograd {k} :=
  match k return let M := 'M[R]_(exp2 k) in M -> M -> M with
  | 0%N => fun A B => A *m B
  | l.+1 => fun A B =>
    if l <= C then A *m B else
    let A11 := ulsubmx A in
    ...
    let B22 := drsubmx B in
    let X := A11 - A21 in
    let Y := B22 - B12 in
    let C21 := winograd X Y in
    ...
    block_mx C11 C12 C21 C22
  end.

Lemma winogradP : forall n (M N : 'M[R]_(exp2 n)),
  (winograd M N) = (M *m N).
```

```
Fixpoint winogradI k :=
  match k return let M := seqmatrix R in M -> M -> M with
  | 0%N => fun A B => mulseqmx A B
  | l.+1 => fun A B =>
    if l <= C then mulseqmx A B else
    let off := exp2 l in
    let A11 := ulsubseqmx off off A in
    ...
    let B22 := drsubseqmx off off B in
    let X := subseqmx A11 A21 in
    let Y := subseqmx B22 B12 in
    let C21 := winogradI l X Y in
    ...
    block_seqmx C11 C12 C21 C22
  end.

Lemma winogradE :
 {morph (@seqmx_of_mx _ (exp2 k) (exp2 k)) :
    M N / winograd M N >-> winogradI k M N}.
```

Benchmarks


Naive product
Fast product
Time (s)
600
400
200
0
0
500
1,000
1,500
2,000
Size

Coq's arrays

- Implemented in a branch based on Coq's trunk
- Arrays are terms of a specific opaque type
- Equiped with operators and axioms
- Using persistent arrays (functional interface, imperative implementation)

Major improvements:

- Constant time access
- Efficient memory representation

Array-based matrices

Array-based row-major representation:

```
Variable R : ringType.
Definition arraymatrix := array (array R).
```

Link with matrices of the library:

```
Definition arraymx_of_mx n =
  match n with
  | O => fun M => make 0 (make 0 default)
  | p.+1 =>
      fun M => let k := of_Z (Z_of_nat n) in
      let row i :=
        let Mi := M (ord_of_int i) in
        init k (fun j => Mi (ord_of_int _ j)) default
      in
      init k row (make 0 default)
  end.
```

Array size is limited, so lemmas express size constraints:

```
Hypothesis Hn0 : n <= max_array_length.

Lemma int_of_ordK : cancel (@int_of_ord n) ord_of_int.
Lemma ord_of_intK :
  {in [pred i| (i < of_Z (Z_of_nat n))%int31],
  cancel ord_of_int (@int_of_ord n)}.
```

Relaxed determinant implementation

- Matrix as sequences of sequences contain too little dimension information
- Extra argument for the dimension
- Expansion along the first column
  - Internal recursion for the expansion function
  - Use `behead` to remove first elements of each row
  - Use `rcons` to add an element at the end of sequence

Determinant code

```
Fixpoint sdet n (mat : seq (seq R)) : R :=
  if n is n'.+1 then
    let fix expand upper lower :=
      if lower is (a :: v) :: lower' then
        let minor := sdet n' (upper ++ map behead lower') in
        a * minor - expand (rcons upper v) lower'
      else 0
    in expand [::] mat
  else 1.
Lemma sdet_correct n M :
  size M <= n -> sdet n M = \det (mx_of_seqmx n n M).
Proof.
  (* 32 line proof *)
```

Determinant code (separating expand)

- expand is unaware of the matrix dimension

```
Fixpoint expand sign (mat : seq (seq R)) det acc : R :=
  if mat is a :: tl then
    sign * head 0 a * det (rev acc ++ (map behead tl)) +
    expand (- sign) tl det (behead a :: acc)
  else
    0.
```

- expand is unaware of the matrix dimension

```
Fixpoint sdet n (mat : seq (seq R)) : R :=
  if n is p.+1 then expand 1 mat (sdet p) [::] else 1.
```

Separate statement for expand

```
Lemma expand_correct :
  forall mat p k l d s acc v (h :(p.+1 = k + l)%N),
   size acc = k -> size mat <= l -> size v = k ->
    (forall mat', size mat' <= p ->
           d mat' = \det (mx_of_seqmx p p mat')) ->
    let r := mx_of_seqmx (k+l) p.+1 (rev (czip v acc) ++ mat)
         in
    expand s mat d acc =
      s * (-1)^+ k * \sum_(i < l) r (rshift k i) 0 *
      cofactor (castmx (refl_equal (k + l)%N, h) r)
         (rshift k i) (cast_ord h ord0).
Proof.
  (* 48 line proof *)
Qed.
```

Failed attempt

```
Lemma expand_correct :
 forall mat p k l d s acc (v:'M_(k,1))(h:(p.+1= k + l)%N),
   size acc = k -> size mat = l ->
    (forall mat', size mat' <= p ->
       d mat' = \det (mx_of_seqmx p p mat')) ->
    let ref := col_mx (row_mx v (mx_of_seqmx k p (rev acc)))
                (mx_of_seqmx l p.+1 mat) in
    expand s mat d acc =
      s * (-1)^+ k * \sum_(i < l) ref (rshift k i) 0 *
      cofactor (castmx ((refl_equal (k + l)%nat), h) ref)
         (rshift k i) (cast_ord h ord0).
```

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- General approach to organize execution-oriented and properties-oriented definitions
- The usual pitfall: trying to prove correctness on a concrete implementation
- We make heavy use of the computational capabilities of Coq
- We have short proofs (4 lines for partial Winograd)
- Realistic size problems can already be treated (4096×4096 dense matrices)

Thank you !