Towards a certified computation of homology groups for digital images*

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4th International Workshop on Computational Topology in Image Context (CTIC 2012)

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R. Mikhailov and J. Wu. On homotopy groups of the suspended classifying spaces. Algebraic and Geometric Topology 10(2010), 565 – 625.

- Computable aim (without computers!)
- Theorem 5.4: Let A_4 be the 4-th alternating group. Then $\pi_4(\Sigma K(A_4, 1)) = \mathbb{Z}_4$

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In that situation

Analyze correctness of programs to ensure correctness of mathematical results

Goal

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Certified computations of homology groups from biomedical digital images

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- Need of trustworthy tools

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Certified computations of homology groups from biomedical digital images

- Need of trustworthy tools
- Formally verified using an interactive proof assistant

Interactive Proof Assistants

- What is an Interactive Proof Assistant?
  - Software tool for the development of formal proofs
  - Man-Machine collaboration:
    - Human: design the proofs
    - Machine: fill the gaps
  - Examples: Isabelle, Hol, ACL2, Coq, ...

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  - Examples: Isabelle, Hol, ACL2, Coq, . . .
- Applications:
  - Mathematical proofs:
    - Four Color Theorem
    - Fundamental Theorem of Algebra
    - Kepler conjecture
  - Software and Hardware verification:
    - C compiler
    - AMD5K86 microprocessor
    - . . .

Coq/SSReflect

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  - Based on Calculus of Inductive Constructions
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- Demo

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  - Algebra
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General Goal

An Algebraic Topology library formalized in COQ/SSREFLECT

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Verification in Coq/SSReflect

Algebraic Topology notions already formalized in Coq/SSReflect:

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Algebraic Topology notions already formalized in Coq/SSReflect:

Simplicial Complexes:

```
(* V is the vertex set *)
Variable V : finType.

(* A simplex over V is a finite subset of V *)
Definition simplex := {set V}.

(* Simplicial Complex *)
Definition simplicial_complex (c : {set simplex}) :=
   forall x, x \in c -> forall y : simplex, y \subset x -> y \in c.

(* Generation of simplicial complex from a list of simplices *)

Definition create_sc (s : seq simplex) : {set simplex} :=
  \bigcup_(sp <- s) powerset sp.

Lemma create_sc_correct : forall s, simplicial_complex (create_sc s).
```

Verification in Coq/SSReflect

Algebraic Topology notions already formalized in Coq/SSReflect:

Boundary Matrices of Simplicial Complexes:

```
Lemma incidence_matrices_sc_product:
  forall (V:finType) (n:nat) (sc: {set (simplex V)}),
    simplicial_complex sc ->
       (incidence_mx_n sc n) *m (incidence_mx_n sc (n.+1)) = 0.
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Homology:

```
Variable (K : fieldType) (V1 V2 V3 : vectType K)
         (f : linearApp V1 V2) (g : linearApp V2 V3).

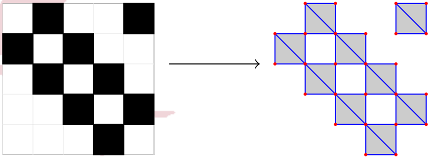
Definition Homology := ((lker g) :\: (limg f)).

Definition dim_homology (mxf:'M[K]_(v1,v2)) (mxg:'M[K]_(v2,v3)) :=
  v2 - \rank mxg - \rank mxf.

Lemma dimHomologyrankE: mxf *m mxg = 0 ->
  \dim Homology (LinearApp mxf) (LinearApp mxg) = dim_homology mxf mxg.
```

Verification in Coq/SSReflect

Connection with digital topology:



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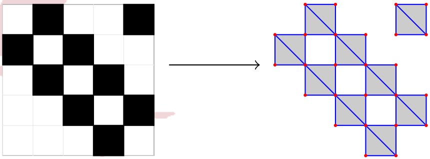
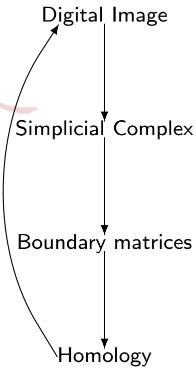


- An image is represented by means of a list of lists of booleans
- Function `createfacets` generates a list of simplexes from an image
- Correctness of `createfacets` has been proved

Verification in Coq/SSReflect

Putting all the pieces together:



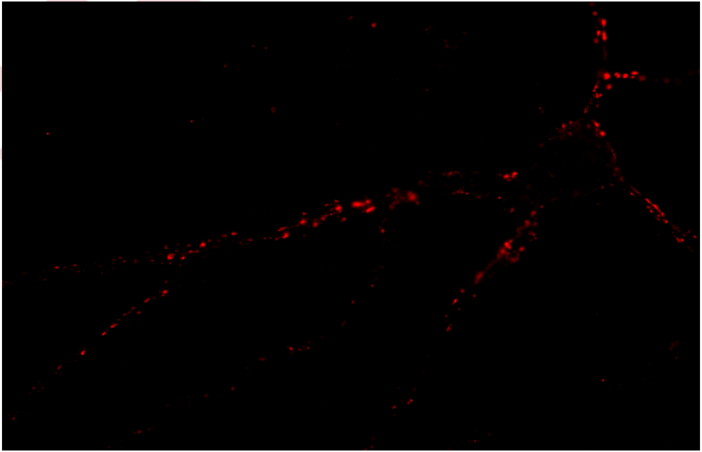
Achievement

Certified computation of Homology from digital images

Testing our framework: Counting synapses

- Synapses are the points of connection between neurons
- Relevance: Computational capabilities of the brain
- Procedures to modify the synaptic density may be an important asset in the treatment of neurological diseases
- An automated and reliable method is necessary

Testing our framework: Counting synapses



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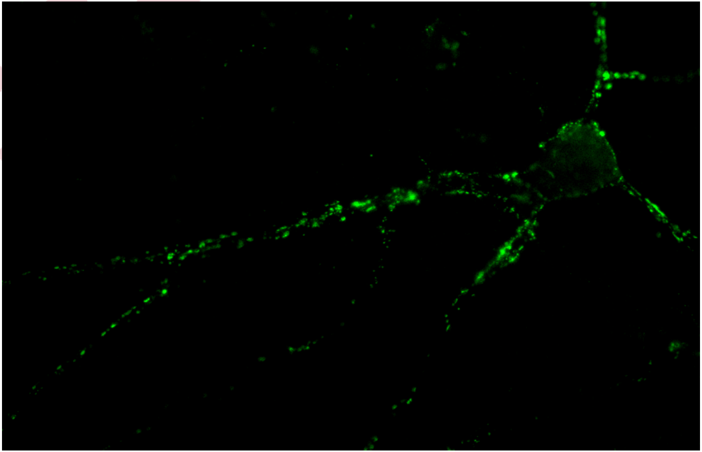

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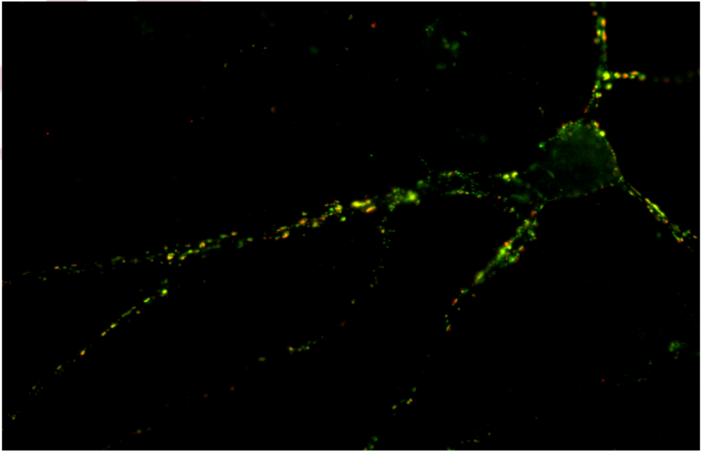

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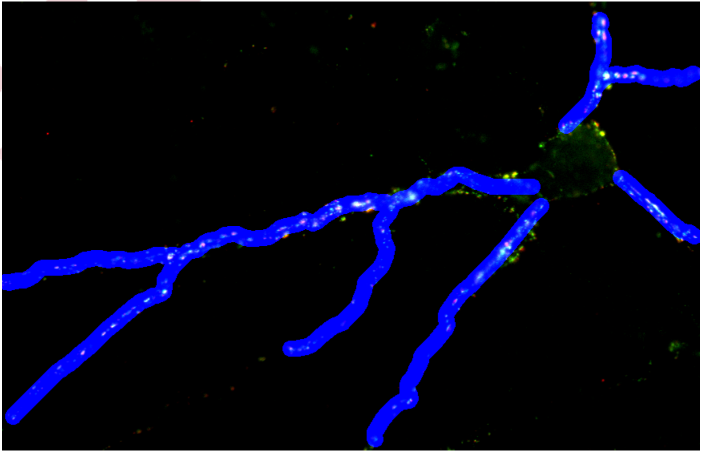

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- Counting synapses:
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  - SynapCountJ: software to measure synaptic density evolution

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- Achieving a better efficiency:
  - Extraction mechanism
  - Internal compilation of Coq terms to OCaml
  - Sparse matrices
  - Better representations for images
  - Reduction of matrices keeping the same homological information

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Discrete Vector Fields + Effective Homology

Reduction process:

- (Algebraic setting of) Discrete Morse Theory

A. Romero and F. Sergeraert. Discrete Vector Fields and Fundamental Algebraic Topology. 2010. http://arxiv.org/abs/1005.5685v1

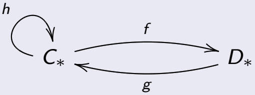
- Effective Homology

J. Rubio and F. Sergeraert. Constructive Algebraic Topology. Bulletin des Sciences Mathématiques, 2002, vol. 126, pp. 389-412

Effective Homology

Definition

A reduction ρ between two chain complexes C_* y D_* (denoted by $\rho : C_* \Rightarrow D_*$) is a triple $\rho = (f, g, h)$



satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

Theorem

If $C_* \Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

Discrete Vector Fields

Definition

Let $C_* = (C_p, d_p)_{p\in\mathbb{Z}}$ *a free chain complex with distinguished* $\mathbb{Z}$*-basis* $\beta_p \subset C_p$*. A discrete vector field* V *on* C_* *is a collection of pairs* $V = \{(\sigma_i; \tau_i)\}_{i\in I}$ *satisfying the conditions:*

- *Every* σ_i *is some element of* β_p*, in which case* $\tau_i \in \beta_{p+1}$*. The degree p depends on i and in general is not constant.*
- *Every component* σ_i *is a regular face of the corresponding* τ_i*.*
- *Each generator (cell) of* C_* *appears at most one time in* V*.*

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- *Every component σ_i is a regular face of the corresponding τ_i.*
- *Each generator (cell) of C_* appears at most one time in V.*

Definition

A V-path of degree p and length m is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ satisfying:

- *Every pair $(\sigma_{i_k}, \tau_{i_k})$ is a component of V and τ_{i_k} is a p-cell.*
- *For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$.*

Discrete Vector Fields

Definition

A discrete vector field V is admissible if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every V-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

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Theorem

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ be an admissible discrete vector field on C_*. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C_p^c, d_p')$ where $C_p^c = \mathbb{Z}[\beta_p^c]$ is the free $\mathbb{Z}$-module generated by the critical p-cells.

Vector fields and integer matrices

M is one of the boundary matrices of the chain complex:

$$\ldots \leftarrow \mathbb{Z}^m \stackrel{M}{\longleftarrow} \mathbb{Z}^n \leftarrow \ldots$$

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Definition

An admissible vector field V for M is nothing but a set of integer pairs $\{(a_i, b_i)\}$ satisfying these conditions:

1. $1 \leq a_i \leq m$ *and* $1 \leq b_i \leq n$
2. *The entry* $M[a_i, b_i]$ *of the matrix is* ± 1
3. *The indices* a_i *(resp.* b_i*) are pairwise different*
4. *Non existence of loops*

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```
Definition admissible_dvf (M: 'M['F_2]_(m,n)) (V: seq ('I_m * 'I_n)) (
    ords : simpl_rel 'I_m) :=
  all [pred p | M p.1 p.2 == 1] V &&
  uniq (map (@fst _ _) V) && uniq (map (@snd _ _) V) &&
  all [pred i | ~~ (connect ords i i)] (map (@fst _ _) V).
```

Romero-Sergeraert algorithms

Algorithm (Romero-Sergeraert 2010)

Input: A matrix M
Output: An admissible discrete vector field for M

- Implemented with the function `gen_adm_dvf`
- Verified in COQ/SSREFLECT

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Lemma admissible_gen_adm_dvf m n (M : 'M['F_2]_(m,n)) :
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Algorithm (Romero-Sergeraert 2010)

Input: A matrix M and an admissible discrete vector field of M
Output: A reduced matrix M'

- Already implemented
- Verification in Coq/SSReflect is an ongoing work

Results

Some remarkable results have been obtained:

- 500 randomly generated images:
  - Initial size of matrices: $\sim 100 \times 300 \to \sim 12$ seconds
  - After reduction process: $\sim 5 \times 50 \to$ milliseconds
  - Most of reduced matrices were null

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  - Most of reduced matrices were null
- Biomedical images:
  - Initial size of matrices: $\sim 700 \times 1400 \rightarrow \infty$
  - After reduction process just 25 seconds

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Conclusions

- Towards an Algebraic Topology Formal library

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- Application to a biomedical problem: counting synapses

Further work

- Formalization aspects:
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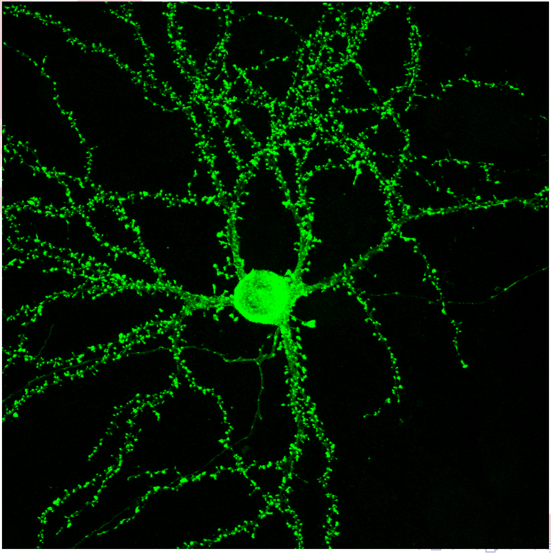
- Formalization aspects:
  - Correctness of reduction process
  - Integer homology computation
- Efficiency issues:
  - Better representations and more efficient algorithms
- Homology certified programs applied to more interesting biomedical cases:
  - Homology group in dimension 1 (structure detection)
  - Persistent Homology (denoising)

Further work



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