Computability of Homology groups

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Abstract. Some notes about the computability of homology groups of chain complexes. These notes are based on [3,6,5,7,1,4,2].

1 Group theory

Theorem 1 (First Isomorphism Theorem). *Let G and H be groups, and let $\varphi : G \to H$ be a surjective homomorphism, then $H \simeq G/\ker\varphi$.*

The above theorem corresponds with `first_isog` SSReflect theorem.

Theorem 2. *Let G be a group and $N \trianglelefteq G$. Then, the map*

$$\begin{aligned} \pi : G &\to G/N \\ g &\mapsto gN \end{aligned}$$

is a surjective homomorphism. Moreover, $\ker\pi = N$.

The above theorem corresponds with `ker_coset` SSReflect theorem.

Definition 1. *The* direct product *of several groups $G_1, \ldots, G_n$ is the Cartesian product endowed with an operation defined elementwise. The direct sum of the groups $G_1, \ldots, G_n$ is denoted by $G_1 \times \ldots \times G_n$.*

When the groups involved are abelian and written with additive notation, it is common to use the terminology direct sum *instead of direct product and use the notation $G \oplus H$ instead of $G \times H$.*

Lemma 1 (Lemma 3.6.1 of [3]). *If H_i is a subgroup of an Abelian group G_i, then*

$$\frac{G_1 \oplus G_2}{H_1 \oplus H_2} \simeq \frac{G_1}{H_1} \oplus \frac{G_2}{H_2}.$$

2 Free Abelian Groups

Definition 2. *Let G be an Abelian group, a subset S of G is* linearly independent *if whenever $x_1, \ldots, x_n$ are distinct elements of S and $r_1, \ldots, r_n$ are elements in $\mathbb{Z}$, if*

$$r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0,$$

then $r_i = 0$ for all i.

A basis *for G is a linearly independent set $S = \{x_1, \ldots, x_n\}$ such that each $g \in G$ can be written uniquely as a finite sum*

$$g = \sum_{i=1}^{n} r_i x_i, \quad r_i \in \mathbb{Z}$$

An abelian group G is free *if it has a basis.*

Proposition 1 (Proposition 3.5.2 of [3]). *Let G be an Abelian group and let $x_1, \ldots, x_n$ be distinct nonzero elements of G. The following conditions are equivalent:*

1. *The set $S = \{x_1, \ldots, x_n\}$ is a basis of G.*
2. *The map*
$$(r_1, \ldots, r_n) \mapsto r_1 x_1 + r_2 x_2 + \cdots + r_n x_n$$
is an isomorphism from $\mathbb{Z}^n$ to G.
3. *For each i, the map $r \mapsto r x_i$ is injective, and $M = \mathbb{Z}x_1 \oplus \ldots \oplus \mathbb{Z}x_n$.*

Definition 3. *Let G be an Abelian group and S a subset of G; then, the* subgroup generated by S *is*

$$\mathbb{Z}S := \{n_1 x_1 + n_2 x_2 + \cdots + n_d x_d \mid d \geq 0, n_i \in \mathbb{Z}, \text{ and } x_i \in S\}.$$

An abelian group is said to be finitely generated *if it is generated by a finite subset.*

Remark 1. Every finite group is finitely generated.

Definition 4. *Let G be an Abelian group. An element $g \in G$ has* finite order *if $ng = 0$ for some positive integer n. The set of all elements of finite order in G is a subgroup T of G, called the* torsion subgroup. *We say that G is a* torsion group *if $G = T$. If T vanishes, we say G is* torsion free.

Remark 2. G/T is torsion free.

Remark 3. A free abelian group is necessarily torsion free.

Proposition 2 (Proposition 3.5.5 of [3]). *Any two bases of a finitely generated free Abelian group have the same cardinality.*

Definition 5. *The* rank *of a finitely generated free abelian group is the cardinal of any basis.*

Proposition 3 (Grushko theorem). *Let G and H be finitely generated groups. Then $rank(G \oplus H) = rank(G) + rank(H)$.*

is `p_rank_dprod` SSReflect theorem?.

Proposition 4 (Corollary 3.5.8 of [3]). *Every subgroup of a finitely generated abelian group is finitely generated.*

Proposition 5. *Any quotient of a finitely generated Abelian group is finitely generated Abelian (simply take the images of the generators in the quotient).*

Theorem 3 (Corollary 10.16 of [7]). *Let G be an Abelian Group and H be a subgroup G such that G/H is free abelian, then H is a direct summand of G - that is, $G = H \oplus K$ where $K \leqslant G$ and $K \simeq G/H$.*

Theorem 4 (Theorem 8.5 of [5]). *Let G be a finitely generated Abelian group, and let T be the torsion subgroup of G. Then T is finite and G/T is free.*

Lemma 2 (Lemma 11.1 of [6]). *Let G be a free Abelian group of rank n. If H is a subgroup of G, then H is free of rank $r \leq n$.*

Lemma 3 (Lemma 11.2 of [6]). *If G is a free abelian group, any subgroup H of G is free.*

Theorem 5 (Theorem 11.3 of [6]). *Let G and H be free abelian groups of ranks n and m respectively; let $f : G \to H$ be a homomorphism. Then there are basis for G and H such that, relative to these basis, the matrix of f has the form*

$$B = \left(\begin{array}{ccc|c} b_1 & & & \\ & \ddots & & 0 \\ & & b_k & \\ \hline & 0 & & 0 \end{array}\right)$$

where $b_i \geq 1$ and $b_1|b_2|\ldots|b_k$.

Theorem 6 (Theorem 4.2 of [6], Theorem 3.5.13 of [3]). *Let F be a free Abelian group whose rank is n and R be a subgroup of F; then there is a basis $e_1, \ldots, e_n$ for F and integers $t_1, \ldots, t_k$ such that*

1. *$t_1e_1, \ldots t_ke_k$ is a basis for R.*
2. *$t_1|t_2|\ldots|t_k$.*

Theorem 7 (The fundamental theorem of finitely generated abelian groups, Theorem 4.3 of [6], Theorem 3.6.2 of [3]). *Let G be a finitely generated abelian group. Let T be its torsion subgroup; then,*

1. *there is a free abelian subgroup H of G having finite rank β such that $G = H \oplus T$.*
2. *There are finite cyclic groups $T_1, \ldots, T_k$ where T_i has order $t_i > 1$ such that $t_1|t_2|\ldots|t_k$ and*
$$T = T_1 \oplus \ldots \oplus T_k.$$
3. *The numbers β and $t_1, \ldots, t_k$ are uniquely determined by G.*

3 Homology

Definition 6. *A* chain complex $\mathcal{C}$ *is a sequence*

$$\cdots \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \cdots$$

of abelian groups C_i and homomorphisms ∂_i, indexed on the integers, such that $\partial_p \circ \partial_{p+1} = 0 \ \forall p$.

The p-th homology group *of $\mathcal{C}$ is defined by the equation*

$$H_p(\mathcal{C}) = \ker \partial_p / im\ \partial_{p+1}.$$

From now on, we will denote $\ker \partial_p$ and im ∂_{p+1} by Z_p and B_p respectively. In addition, we will consider a chain complex $\mathcal{C}$ where each group C_p is free of finite rank.

Definition 7. *Let W_p consist of all elements c_p of C_p such that some non-zero multiple of c_p belongs to B_p - that is,*

$$W_p = \{c_p \in C_p | \exists \lambda \in \mathbb{Z} \setminus \{0\}, \lambda c_p \in B_p\}.$$

This group is called the group of weak boundaries.

Lemma 4.

$$B_p \leqslant W_p \leqslant Z_p \leqslant C_p$$

Proof.

$Z_p \leqslant C_p$**.** It comes from the definition of Z_p.
$B_p \leqslant W_p$**.** It is trivial, just take $\lambda = 1$.
$W_p \leqslant Z_p$**.** C_p is free, then C_p is torsion free (remark 3). So, $\lambda c_p \neq 0 \ \forall \lambda \neq 0$ and $c_p \neq 0$. Using the definition of W_p, $\forall c_p \in W_p \ \exists \lambda \neq 0$ such that $\lambda c_p \in B_p$. Now, as $\partial_p \partial_{p+1} = 0$ then $\partial_p(\lambda c_p) = 0$ implies that $\lambda \partial_p(c_p) = 0$, then, $\partial_p(c_p) = 0$. Therefore, $\lambda c_p \in \ker \partial_p = Z_p$.

Lemma 5 (Decomposition of Z_p). *W_p is a direct summand of Z_p - that is, there exist a subgroup V_p of Z_p such that*

$$Z_p = V_p \oplus W_p.$$

Proof. First of all, let us note that $H_p = Z_p/B_p$ is a finitely generated abelian group (Propositions 4 and 5), then it can be written as the direct sum $F_p \oplus T_p$ where F_p is a free subgroup of H_p and T_p is the torsion subgroup of H_p (Theorem 7 (1)).

Consider now, the natural projection q from Z_p to H_p/T_p. This projection can be seen as the composition of the 2 projections q_1 and q_2: $q = q_2 \circ q_1$ such that:

$$\begin{array}{ccccc} Z_p & \xrightarrow{q_1} & H_p = Z_p/B_p & \xrightarrow{q_2} & H_p/T_p \\ c_p & \mapsto & \overline{c_p} & \mapsto & \widetilde{c_p} \end{array}$$

The kernel of this projection is $\ker(q) = \{c_p \in Z_p | q(c_p) = 0\}$.

Let us note that $\forall c_p \in \ker(q)$, $\overline{c_p} \subseteq T$ (use Theorem 2). As T_p is a torsion group, $\overline{c_p}$ is an element of finite order (Definition 4) in H_p. Then, there exist $\lambda \in \mathbb{Z} \setminus \{0\}$ such that $\lambda \overline{c_p} = \overline{0_p} = B_p$. That is, there is a $b_p \in B_p$ such that $\lambda c_p = b_p$. Therefore,

$$\ker(q) = \{c_p \in Z_p | \exists \lambda \neq 0, \lambda c_p in B_p\} = W_p.$$

Using now Theorem 1, we obtain $Z_p/W_p \simeq H_p/T_p$. H_p/T_p is finitely generated and torsion free. Then, using Theorem 4, H_p/T_p is free; so, Z_p/W_p is free. Now, by Theorem 3, $Z_p = V_p \oplus W_p \simeq Z_p/W_p \oplus W_p$.

Corollary 1. *$H_p \simeq Z_p/W_p \oplus W_p/B_p$.*

Proof. $H_p = Z_p/B_p = (V_p \oplus W_p)/B_p \simeq V_p \oplus W_p/B_p \simeq Z_p/W_p \oplus W_p/B_p$.

Remark 4. Z_p/W_p is free and W_p/B_p is a torsion group.

Lemma 6. *Let $\{e_1^p, \ldots, e_{n_p}^p\}$ and $\{e_1^{p-1}, \ldots, e_{n_{p-1}}^{p-1}\}$ be basis of C_p and C_{p-1} respectively; such that the matrix of C_p relative to these basis has the normal form:*

$$\begin{array}{c|ccc|ccc|} & e_1^p & \cdots & e_{k_p}^p & e_{k_p+1}^p & \cdots & e_{n_p}^p \\ e_1^{p-1} & b_1^p & & & & & \\ \vdots & & \ddots & & & 0 & \\ e_{k_p}^{p-1} & & & b_{k_p}^p & & & \\ \hline e_{k_p+1}^{p-1} & & & & & & \\ \vdots & & 0 & & & 0 & \\ e_{n_{p-1}}^{p-1} & & & & & & \end{array}$$

Then, the following hold:

1. *$e_{k_p+1}^p, \ldots, e_{n_p}^p$ is a basis for Z_p.*
2. *$b_1^p e_1^{p-1}, \ldots, b_{k_p}^p e_{k_p}^{p-1}$ is a basis for B_{p-1}.*
3. *$e_1^{p-1}, \ldots, e_{k_p}^{p-1}$ is a basis for W_{p-1}.*

Proof. Let $c_p \in C_p$ - that is, $c_p = \sum_{i=1}^{n_p} a_i e_i^p$, then $\partial_p(c_p) = \sum_{i=1}^{k_p} b_i^p a_i e_i^{p-1}$.

1. $\partial_p(c_p) = 0 \Leftrightarrow \forall i = 1, \ldots, k_p \; a_i = 0$. Then, a basis for Z_p is $e_{k_p+1}^p, \ldots, e_{n_p}^p$.

2. $\forall c_{p-1} \in B_{p-1}$, $\exists c_p \in C_p$ such that $\partial_p(c_p) = c_{p-1}$. Then, there exists $\{a_i\}_{i\in\{1,\ldots,k_p\}}$ such that

$$c_{p-1} = \sum_{i=1}^{k_p} b_i^p a_i e_i^{p-1}.$$

And, $\forall i \in \{1,\ldots,k_p\}$, $b_i^p \neq 0$; therefore, $b_1^p e_1^{p-1}, \ldots, b_{k_p}^p e_{k_p}^{p-1}$ is a basis of B_{p-1}.

3. $W_{p-1} = \{c_{p-1} \in C_{p-1} | \exists \lambda \in \mathbb{Z} \setminus \{0\}, \lambda c_{p-1} \in B_{p-1}\}$. By (2), $\forall i \in \{1,\ldots,k_p\}$ $b_i^p e_i^{p-1} \in B_{p-1}$; then, $\{e_i^{p-1}\}_{i\in\{1,\ldots,k_p\}} \subseteq W_{p-1}$.

   Conversely, let $c_{p-1} = \sum_{i=1}^{n_{p-1}} a_i' e_i^{p-1} \in W_{p-1}$. Then, $\exists \lambda \neq 0$, $\exists c_p \in C_p$ such that $\lambda c_{p-1} = \partial_p(c_p) \in B_{p-1}$. So,

$$\lambda \sum_{i=1}^{n_{p-1}} a_i' e_i^{p-1} = \sum_{i=1}^{k_p} b_i^p a_i e_i^{p-1}.$$

   Then, $a_i' = 0$ $\forall i \in \{k_p+1,\ldots,n_{p-1}\}$; therefore, $e_1^{p-1},\ldots,e_{k_p}^{p-1}$ is a finite set of generators of W_p. In addition, as these elements appear in the basis of Z_p, they are linearly independent; so, they constitute a basis of W_{p-1}.

Theorem 8. $H_p(\mathcal{C}) \simeq \mathbb{Z}^{n_p-k_p-k_{p+1}} \oplus \mathbb{Z}/b_1^{p+1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/b_{k_{p+1}}^{p+1}\mathbb{Z}$

Proof. $H_p(\mathcal{C}) \simeq Z_p/W_p \oplus W_p/B_p$.

First of all, let us see that $Z_p/W_p \simeq \mathbb{Z}^{n_p-k_p-k_{p+1}}$. Z_p/W_p is free; so, $Z_p/W_p \simeq \mathbb{Z}^d$ where d is the rank of Z_p/W_p.

Applying Proposition 3, $rank(Z_p) = rank(Z_p/W_p) + rank(W_p)$; then,

$$rank(Z_p/W_p) = rank(Z_p) - rank(W_p).$$

Using Lemma 6, $rank(Z_p) = \sharp|\{e_{k_p+1}^p,\ldots,e_{n_p}^p\}| = n_p - k_p$; and $rank(W_p) = \sharp|\{e_1^p,\ldots,e_{k_{p+1}}^p\}| = k_{p+1}$. Then $rank(Z_p/W_p) = n_p - k_p - k_{p+1}$; so,

$$Z_p/W_p \simeq \mathbb{Z}^{n_p-k_p-k_{p+1}}$$

Now, let us prove $B_p/W_p \simeq \mathbb{Z}/b_1^{p+1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/b_{k_{p+1}}^{p+1}\mathbb{Z}$.

$$\begin{aligned} W_p/B_p = &\ \frac{e_1^{p+1}\mathbb{Z}\oplus\ldots\oplus e_{k_{p+1}}^{p+1}\mathbb{Z}}{b_1^{p+1}e_1^{p+1}\mathbb{Z}\oplus\ldots\oplus b_{k_{p+1}}^{p+1}e_{k_{p+1}}^{p+1}\mathbb{Z}} \\ \simeq &\ \frac{e_1^{p+1}\mathbb{Z}}{b_1^{p+1}e_1^{p+1}\mathbb{Z}} \oplus \ldots \oplus \frac{e_{k_{p+1}}^{p+1}\mathbb{Z}}{b_{k_{p+1}}^{p+1}e_{k_{p+1}}^{p+1}\mathbb{Z}} \\ \simeq &\ \mathbb{Z}/b_1^{p+1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/b_{k_{p+1}}^{p+1}\mathbb{Z} \end{aligned}$$

Observe that $v\mathbb{Z} \simeq dv\mathbb{Z}$ since $r \mapsto rv + dv\mathbb{Z}$ is a surjective $\mathbb{Z}$-module homomorphism with kernel $d\mathbb{Z}$.

Therefore,

$$H_p(\mathcal{C}) \simeq \mathbb{Z}^{n_p-k_p-k_{p+1}} \oplus \mathbb{Z}/b_1^{p+1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/b_{k_{p+1}}^{p+1}\mathbb{Z}.$$

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