Incidence Matrices of Simplicial Complex in SSreflect[1]

Jónathan Heras and María Poza

University of La Rioja

September 27, 2010

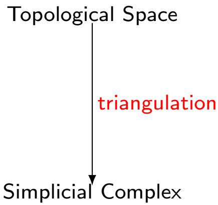
[1] Supported by European Commission FP7, STREP project ForMath

Índice

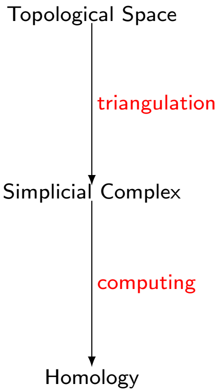
From "General" Topology to Homological Algebra

Topological Space

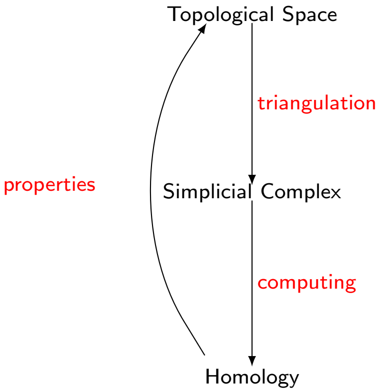
From “General” Topology to Homological Algebra



From “General” Topology to Homological Algebra

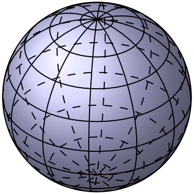


From "General" Topology to Homological Algebra



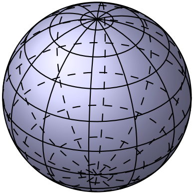
An example

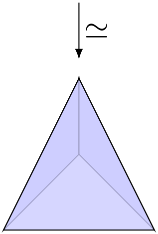
Topological Space



An example

Topological Space





Simplicial Complex:

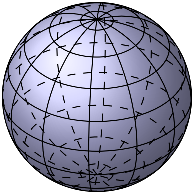
0-simplices: vertices (4 vertices)

1-simplices: edges (6 edges)

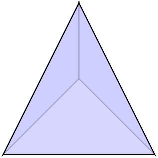
2-simplices: triangles (4 triangles)

An example

Topological Space



$\downarrow \simeq$



Homology groups

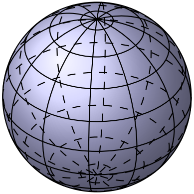
$H_0 = \mathbb{Z}$
$H_1 = 0$
$H_2 = \mathbb{Z}$
$H_3 = 0$
$\ldots$

Simplicial Complex:

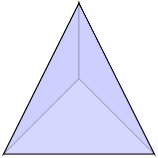
0-simplices: vertices (4 vertices)
1-simplices: edges (6 edges)
2-simplices: triangles (4 triangles)

An example

Topological Space



$\simeq$



Homology groups

$H_0 = \mathbb{Z}$
$H_1 = 0$
$H_2 = \mathbb{Z}$
$H_3 = 0$
$\ldots$

Simplicial Complex:

0-simplices: vertices (4 vertices)
1-simplices: edges (6 edges)
2-simplices: triangles (4 triangles)

Simplicial Complexes

Definition:

Let V be a set, called the vertex set, a *simplex* over V is any finite subset of V.

Simplicial Complexes

Definition:

Let V be a set, called the vertex set, a *simplex* over V is any finite subset of V.

Definition:

Let α and β be simplices over V, we say α *is a face of* β if α is a subset of β.

Simplicial Complexes

Definition:

Let V be a set, called the vertex set, a *simplex* over V is any finite subset of V.

Definition:

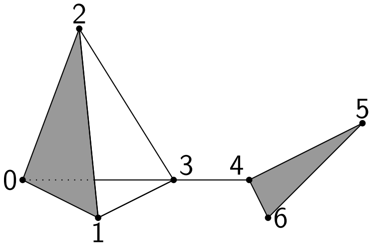
Let α and β be simplices over V, we say α *is a face of* β if α is a subset of β.

Definition:

An *(abstract) simplicial complex* over V is a set of simplices C over V satisfying the property:

$$\forall \alpha \in C, \textit{ if } \beta \subseteq \alpha \Rightarrow \beta \in C$$

Simplicial Complexes

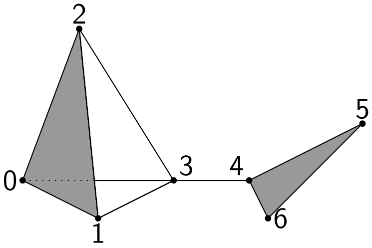


$C = \{\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\},$
$\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{4,5\}, \{4,6\}, \{5,6\},$
$\{0,1,2\}, \{4,5,6\}\}$

Simplicial Complexes

Definition:

The *facets* of a simplicial complex C are the maximal simplices of the simplicial complex.



The facets are: $\{\{1,3\},\{3,4\},\{0,3\},\{2,3\},\{0,1,2\},\{4,5,6\}\}$

Incidence Matrices

Definition

Let X and Y be two enumerated finite sets and r be a relationship between the elements of X and the elements of Y, we call *incidence matrix*

$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

where

$$a_{i,j} = \begin{cases} 1 & \text{si } X[i] \text{ is related to } Y[j] \\ 0 & \text{si } X[i] \text{ is not related to } Y[j] \end{cases}$$

Incidence Matrices of Simplicial Complexes

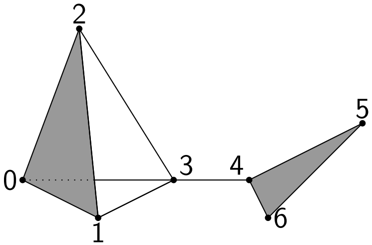
Definition

Let C be a simplicial complex, A the set of n-simplices of C and B the set of $(n-1)$-simplices of C.
We call *incidence matrix* of dimension n $(n \geq 1)$, M_n of the simplicial complex C, to a matrix $p \times q$ where

$$p = \sharp|B| \ \wedge \ q = \sharp|A|$$

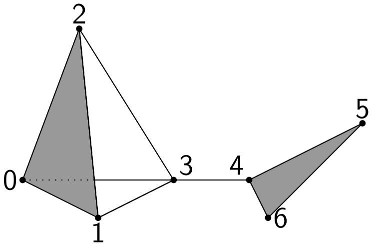
$$M_{i,j} \quad = \quad \begin{cases} 1 & \text{si } B_i \subset A_j \\ 0 & \text{si } B_i \not\subset A_j \end{cases}$$

Incidence Matrices of Simplicial Complexes



	{0, 1}	{0, 2}	{0, 3}	{1, 2}	{1, 3}	{2, 3}	{3, 4}	{4, 5}	{4, 6}	{5, 6}
{0}	1	1	1	0	0	0	0	0	0	0
{1}	1	0	0	1	1	0	0	0	0	0
{2}	0	1	0	1	0	1	0	0	0	0
{3}	0	0	1	0	1	1	1	0	0	0
{4}	0	0	0	0	0	0	1	1	1	0
{5}	0	0	0	0	0	0	0	1	0	1
{6}	0	0	0	0	0	0	0	0	1	1

Incidence Matrices of Simplicial Complexes



	{0, 1, 2}	{4, 5, 6}
{0, 1}	1	0
{0, 2}	1	0
{0, 3}	0	0
{1, 2}	1	0
{1, 3}	0	0
{2, 3}	0	0
{3, 4}	0	0
{4, 5}	0	1
{4, 6}	0	1
{5, 6}	0	1

Incidence Matrices of Simplicial Complexes

Importance of the I.M. of a S.C.

The incidence matrices of simplicial complexes are used to compute the homology of the simplicial complex

Incidence Matrices of Simplicial Complexes

Importance of the I.M. of a S.C.

The incidence matrices of simplicial complexes are used to compute the homology of the simplicial complex

Objective

Facets → Simplicial Complex → Incidence Matrix → Homology

Incidence Matrices of Simplicial Complexes

Importance of the I.M. of a S.C.

The incidence matrices of simplicial complexes are used to compute the homology of the simplicial complex

Objective

Facets → Simplicial Complex → Incidence Matrix → Homology

Incidence Matrices of Simplicial Complexes

Importance of the I.M. of a S.C.

The incidence matrices of simplicial complexes are used to compute the homology of the simplicial complex

Objective

Facets → Simplicial Complex → Incidence Matrix → Homology

Incidence Matrices of Simplicial Complexes

Importance of the I.M. of a S.C.

The incidence matrices of simplicial complexes are used to compute the homology of the simplicial complex

Objective

Facets → Simplicial Complex → Incidence Matrix → Homology

Problem

Theorem: Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Let C be a simplicial complex and n a number natural such that $n \geq 2$, then the product of the incidence matrix of dimension $n-1$, denoted by M_{n-1}, and the incidence matrix of dimension n, denoted by M_n, is equal to the null matrix.

Problem

Theorem: Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Let C be a simplicial complex and n a number natural such that $n \geq 2$, then the product of the incidence matrix of dimension $n-1$, denoted by M_{n-1}, and the incidence matrix of dimension n, denoted by M_n, is equal to the null matrix.

Sketch of the proof.

- Let C_n be the set of n-simplices of C
- Let C_{n-1} be the set of $(n-1)$-simplices of C
- Let C_{n-2} be the set of $(n-2)$-simplices of C

Problem

Theorem: Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Let C be a simplicial complex and n a number natural such that $n \geq 2$, then the product of the incidence matrix of dimension $n-1$, denoted by M_{n-1}, and the incidence matrix of dimension n, denoted by M_n, is equal to the null matrix.

Sketch of the proof.

- Let C_n be the set of n-simplices of C
- Let C_{n-1} be the set of $(n-1)$-simplices of C
- Let C_{n-2} be the set of $(n-2)$-simplices of C

$$M_{n-1} = \begin{array}{c} C_{n-2}[1] \\ \vdots \\ C_{n-2}[r2] \end{array} \overset{\begin{array}{ccc} C_{n-1}[1] & \cdots & C_{n-1}[r1] \end{array}}{\begin{pmatrix} a_{1,1} & \cdots & a_{1,r1} \\ \vdots & \ddots & \vdots \\ a_{r2,1} & \cdots & a_{r2,r1} \end{pmatrix}} \quad M_n = \begin{array}{c} C_{n-1}[1] \\ \vdots \\ C_{n-1}[r1] \end{array} \overset{\begin{array}{ccc} C_n[1] & \cdots & C_n[r3] \end{array}}{\begin{pmatrix} b_{1,1} & \cdots & b_{1,r1} \\ \vdots & \ddots & \vdots \\ b_{r1,1} & \cdots & b_{r1,r3} \end{pmatrix}}$$

Problem

$$M_{n-1} \times M_n = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,\ j} = \sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j}$$

Problem

$$M_{n-1} \times M_n = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,\ j} = \sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j}$$

we need to prove that

$$\forall i, j, c_{i,\ j} = 0$$

in order to prove that $M_{n-1} \times M_n = 0$

Problem

Lemma

Under the previous conditions, $\forall i, j, c_{i,\ j} = 0$

Problem

Lemma

Under the previous conditions, $\forall i, j, c_{i,\ j} = 0$

Proof.

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} = \sum_{j0 | M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} + \sum_{j0 | M_{n-2}[i] \not\subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} + \sum_{j0 | M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \not\subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j} + \sum_{j0 | M_{n-2}[i] \not\subset M_{n-1}[j0] \wedge M_{n-1}[j0] \not\subset M_n[j]} a_{i,\ j0} \times b_{j0,\ j}$$

Problem

Lemma

Under the previous conditions, $\forall i, j, c_{i,\ j} = 0$

Proof.

$$\begin{aligned}\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} &= \Big(\sum_{j0 \mid M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} 1 \Big) + 0 + 0 + 0 \\ &= \sharp |\{ j0 \mid M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j] \}|\end{aligned}$$

Problem

Lemma

Under the previous conditions, let $T \in C_n$ and $x \in C_{n-2}$ if $x \subset T$ then,

$$\sharp|\{y \in C_{n-1}|(x \subset y) \wedge (y \subset T)\}| = 2$$

Problem

Lemma

Under the previous conditions, let $T \in C_n$ and $x \in C_{n-2}$ if $x \subset T$ then,

$$\sharp|\{y \in C_{n-1}|(x \subset y) \wedge (y \subset T)\}| = 2$$

Sketch of the proof.

- $T \in C_n \Rightarrow T = \{a_0, \ldots, a_n\}$
- $x \in C_{n-2} \wedge x \subset T \Rightarrow x = \{a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n\}$
- $y \in C_{n-1} \wedge y \subset T \Rightarrow y = \{a_0, \ldots, \widehat{a_r}, \ldots, a_n\}$
- $y \in C_{n-1} \wedge y \subset T \wedge x \subset y \Rightarrow y = \{a_0, \ldots, \widehat{a_r}, \ldots, a_n\}$ with r={i,j}

Problem

Lemma

Under the previous conditions, let $T \in C_n$ and $x \in C_{n-2}$ if $x \subset T$ then,

$$\sharp|\{y \in C_{n-1} | (x \subset y) \wedge (y \subset T)\}| = 2$$

Sketch of the proof.

- $T \in C_n \Rightarrow T = \{a_0, \ldots, a_n\}$
- $x \in C_{n-2} \wedge x \subset T \Rightarrow x = \{a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_n\}$
- $y \in C_{n-1} \wedge y \subset T \Rightarrow y = \{a_0, \ldots, \widehat{a_r}, \ldots, a_n\}$
- $y \in C_{n-1} \wedge y \subset T \wedge x \subset y \Rightarrow y = \{a_0, \ldots, \widehat{a_r}, \ldots, a_n\}$ with r={i,j}

Then

$$\sharp|\{y \in C_{n-1} | (x \subset y) \wedge (y \subset T)\}| = 2$$

□