Incidence Simplicial Matrices Formalized in Coq/SSReflect*

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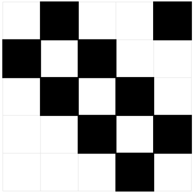
CICM 2011, Calculemus track,
July 22, 2011

*Partially supported by Ministerio de Educación y Ciencia, project MTM2009-13842-C02-01, and by European Commission FP7, STREP project ForMath, n. 243847

Algebraic Topology and Digital Images

Digital Image



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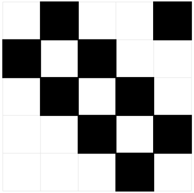
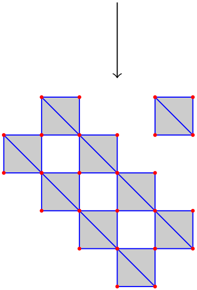
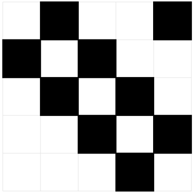


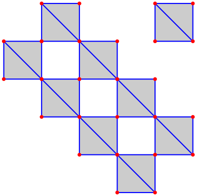


Simplicial complex

Algebraic Topology and Digital Images

Digital Image





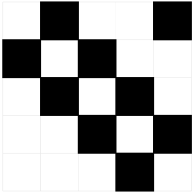
Simplicial complex

$$C_0 = \mathbb{Z}[vertices]$$
$$C_1 = \mathbb{Z}[edges]$$
$$C_2 = \mathbb{Z}[triangles]$$

Chain complex

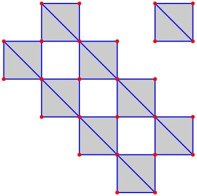
Algebraic Topology and Digital Images

Digital Image



Homology groups

$$H_0 = \mathbb{Z} \oplus \mathbb{Z}$$
$$H_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$



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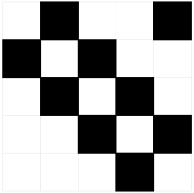

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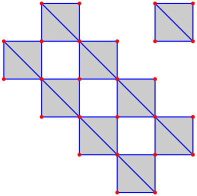

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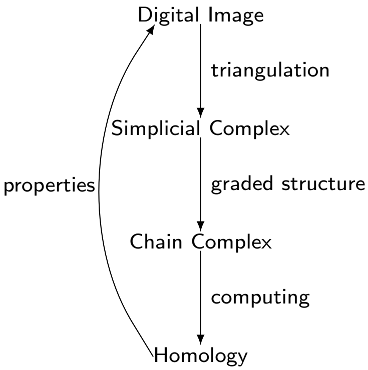
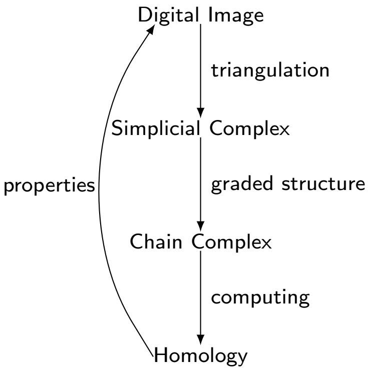
Goal

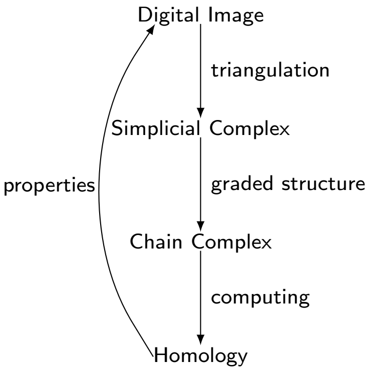

Digital Image
triangulation
Simplicial Complex
properties
graded structure
Chain Complex
computing
Homology

Goal



- Implemented in the Kenzo system

Goal



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General Goal

Formalizing the computation of homology groups of digital images

Goal

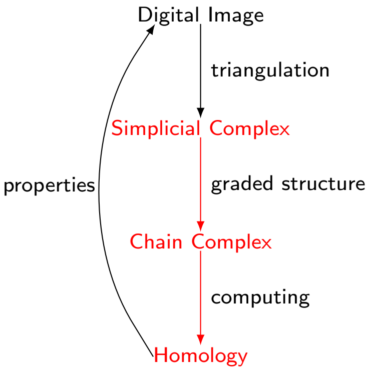

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Table of Contents

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Digital Images

Digital Image → Simplicial Complex → Chain Complex → Homology

- 2D digital images:
  - elements are pixels

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Digital Image → Simplicial Complex → Chain Complex → Homology

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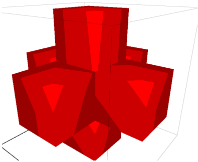
- 3D digital images:
  - elements are voxels



Simplicial Complexes

Digital Image → **Simplicial Complex** → Chain Complex → Homology

Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V

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Definition

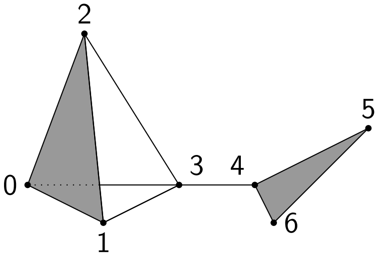
An ordered (abstract) simplicial complex over V is a set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \text{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

Let $\mathcal{K}$ be a simplicial complex. Then the set $S_n(\mathcal{K})$ of n-simplices of $\mathcal{K}$ is the set made of the simplices of cardinality $n+1$

Simplicial Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology



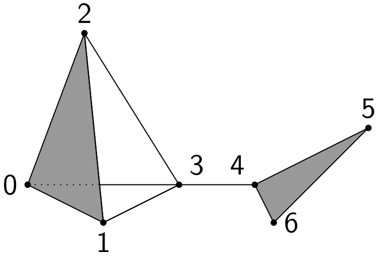
$V = (0, 1, 2, 3, 4, 5, 6)$
$\mathcal{K} = \{\emptyset, (0), (1), (2), (3), (4), (5), (6),$
$(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6),$
$(0, 1, 2), (4, 5, 6)\}$

Simplicial Complexes

Digital Image → **Simplicial Complex** → Chain Complex → Homology

Definition

The facets of a simplicial complex $\mathcal{K}$ are the maximal simplices of the simplicial complex



The facets are: $\{(0,3),(1,3),(2,3),(3,4),(0,1,2),(4,5,6)\}$

Chain Complexes

Digital Image ⟶ Simplicial Complex ⟶ **Chain Complex** ⟶ Homology

Definition

A chain complex C_* is a pair of sequences $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ where:

- *For every $q \in \mathbb{Z}$, the component C_q is an R-module, the chain group of degree q*
- *For every $q \in \mathbb{Z}$, the component d_q is a module morphism $d_q : C_q \rightarrow C_{q-1}$, the differential map*
- *For every $q \in \mathbb{Z}$, the composition $d_q d_{q+1}$ is null: $d_q d_{q+1} = 0$*

Homology

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ **Homology**

Definition

If $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$ is a chain complex:

- *The image $B_q = im\ d_{q+1} \subseteq C_q$ is the (sub)module of q-boundaries*
- *The kernel $Z_q = ker\ d_q \subseteq C_q$ is the (sub)module of q-cycles*

Given a chain complex $C_* = (C_q, d_q)_{q\in\mathbb{Z}}$:

- $d_{q-1} \circ d_q = 0 \Rightarrow B_q \subseteq Z_q$
- Every boundary is a cycle
- The converse is not generally true

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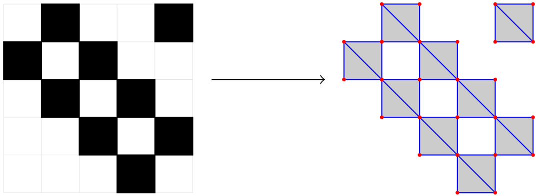
Definition

Let $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ be a chain complex. For each degree $n \in \mathbb{Z}$, the n-homology module of C_* is defined as the quotient module

$$H_n(C_*) = \frac{Z_n}{B_n}$$

From a digital image to a simplicial complex

Digital Image → Simplicial Complex → Chain Complex → Homology



From Simplicial Complexes to Chain Complexes

Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ Homology

Definition

Let $\mathcal{K}$ be an (ordered abstract) simplicial complex. Let $n \geq 1$ and $0 \leq i \leq n$ be two integers n and i. Then the face operator ∂_i^n is the linear map $\partial_i^n : S_n(\mathcal{K}) \to S_{n-1}(\mathcal{K})$ defined by:

$$\partial_i^n((v_0, \ldots, v_n)) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n).$$

The i-th vertex of the simplex is removed, so that an $(n-1)$-simplex is obtained

Definition

Let $\mathcal{K}$ be a simplicial complex. Then the chain complex $C_*(\mathcal{K})$ canonically associated with $\mathcal{K}$ is defined as follows. The chain group $C_n(\mathcal{K})$ is the free $\mathbb{Z}$ module generated by the n-simplices of $\mathcal{K}$. In addition, let $(v_0, \ldots, v_{n-1})$ be a n-simplex of $\mathcal{K}$, the differential of this simplex is defined as:

$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i^n$$

Computing

Digital Image → Simplicial Complex → Chain Complex → Homology

- Computing Homology groups:
  - From a Chain Complex $(C_n, d_n)_{n \in \mathbb{Z}}$:
    - d_n can be expressed as matrices
    - Homology groups are obtained from a diagonalization process

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  - Directly from the Simplicial Complex:
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Digital Image ⟶ Simplicial Complex ⟶ Chain Complex ⟶ Homology

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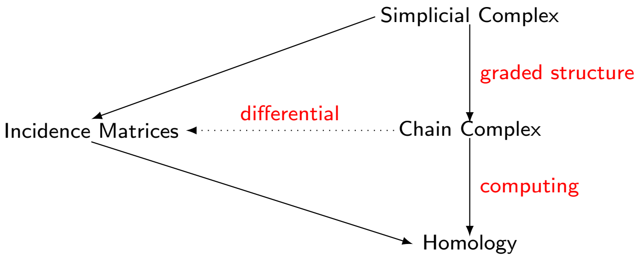
From Simplicial Complexes to Homology



Incidence Matrices

Definition

Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix $m \times n$ where

$$m = \sharp|X| \ \wedge \ n = \sharp|Y|$$

$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

$$a_{i,j} \quad = \quad \begin{cases} 1 & \textit{if } X[i] \textit{ is a face of } Y[j] \\ 0 & \textit{if } X[i] \textit{ is not a face of } Y[j] \end{cases}$$

Incidence Matrices

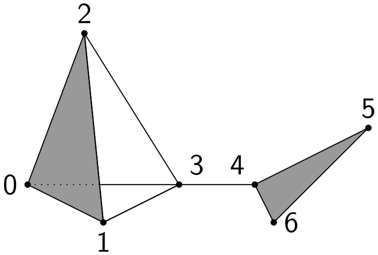
Definition

Let C be a finite set of simplices, A be the set of n-simplices of C with an order between its elements and B the set of $(n-1)$-simplices of C with an order between its elements.
We call incidence matrix of dimension n ($n \geq 1$), to a matrix $p \times q$ where

$$p = \sharp|B| \;\wedge\; q = \sharp|A|$$

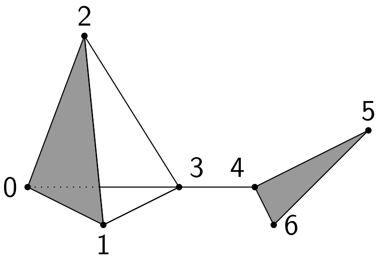
$$M_{i,j} \quad = \quad \begin{cases} 1 & \textit{if } B[i] \textit{ is a face of } A[j] \\ 0 & \textit{if } B[i] \textit{ is not a face of } A[j] \end{cases}$$

Incidence Matrices of Simplicial Complexes



	(0, 1)	(0, 2)	(0, 3)	(1, 2)	(1, 3)	(2, 3)	(3, 4)	(4, 5)	(4, 6)	(5, 6)
(0)	1	1	1	0	0	0	0	0	0	0
(1)	1	0	0	1	1	0	0	0	0	0
(2)	0	1	0	1	0	1	0	0	0	0
(3)	0	0	1	0	1	1	1	0	0	0
(4)	0	0	0	0	0	0	1	1	1	0
(5)	0	0	0	0	0	0	0	1	0	1
(6)	0	0	0	0	0	0	0	0	1	1

Incidence Matrices of Simplicial Complexes



	(0, 1, 2)	(4, 5, 6)
(0, 1)	1	0
(0, 2)	1	0
(0, 3)	0	0
(1, 2)	1	0
(1, 3)	0	0
(2, 3)	0	0
(3, 4)	0	0
(4, 5)	0	1
(4, 6)	0	1
(5, 6)	0	1

Product of two consecutive incidence matrices

Theorem (Product of two consecutive incidence matrices)

Let $\mathcal{K}$ be a finite simplicial complex over V with an order between the simplices of the same dimension and let $n \geq 1$ be a natural number n, then the product of the n-th incidence matrix of $\mathcal{K}$ and the $(n+1)$-incidence matrix of $\mathcal{K}$ over the ring $\mathbb{Z}/2\mathbb{Z}$ is equal to the null matrix

Sketch of the proof

- Let S_{n+1} be the set of $(n+1)$-simplices of $\mathcal{K}$ with an order between its elements
- Let S_n be the set of n-simplices of $\mathcal{K}$ with an order between its elements
- Let S_{n-1} be the set of $(n-1)$-simplices of $\mathcal{K}$ with an order between its elements

Sketch of the proof

- Let S_{n+1} be the set of $(n+1)$-simplices of $\mathcal{K}$ with an order between its elements
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- Let S_{n-1} be the set of $(n-1)$-simplices of $\mathcal{K}$ with an order between its elements

$$M_n(\mathcal{K}) = \begin{array}{c} \\ S_{n-1}[1] \\ \vdots \\ S_{n-1}[r2] \end{array} \begin{array}{c} \begin{matrix} S_n[1] & \cdots & S_n[r1] \end{matrix} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,r1} \\ \vdots & \ddots & \vdots \\ a_{r2,1} & \cdots & a_{r2,r1} \end{pmatrix} \end{array}, \quad M_{n+1}(\mathcal{K}) = \begin{array}{c} \\ S_n[1] \\ \vdots \\ S_n[r1] \end{array} \begin{array}{c} \begin{matrix} S_{n+1}[1] & \cdots & S_{n+1}[r3] \end{matrix} \\ \begin{pmatrix} b_{1,1} & \cdots & b_{1,r1} \\ \vdots & \ddots & \vdots \\ b_{r1,1} & \cdots & b_{r1,r3} \end{pmatrix} \end{array}$$

where $r1 = \sharp|S_n|$, $r2 = \sharp|S_{n-1}|$ and $r3 = \sharp|S_{n+1}|$

Sketch of the proof

$$M_n(\mathcal{K}) \times M_{n+1}(\mathcal{K}) = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,j} = \sum_{1 \leq k \leq r1} a_{i,k} \times b_{k,j}$$

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we need to prove that

$$\forall i, j, c_{i,j} = 0$$

in order to prove that $M_n \times M_{n+1} = 0$

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Since k enumerates the indices of elements of S_n:

$$c_{i,j} = \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \text{ with } F(Y,Z) = \begin{cases} 1 & \text{if } Y \in dZ \\ 0 & \text{otherwise} \end{cases}$$

where

$$dZ = \{Z \setminus \{x\} \mid x \in Z\}$$

Sketch of the proof

$$c_{i,j} = \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j])$$

Sketch of the proof

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\begin{aligned}
c_{i,j} &= \sum_{X \in S_n} F(S_{n-1}[i], X) \times F(X, S_{n+1}[j]) \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 1 \\
&\quad + \sum_{X \in S_n | X \notin \partial S_{n+1}[j]} F(S_{n-1}[i], X) \times 0 \\
&= \sum_{X \in S_n | X \in \partial S_{n+1}[j]} F(S_{n-1}[i], X)
\end{aligned}
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- $S_{n-1}[i] \not\subset S_{n+1}[j]$
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  $\forall x \in S_{n-1}[i], F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 0$
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  $F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 1$

$$
\begin{aligned}
c_{i,j} &= \sum_{x \in S_{n+1}[j] | x \notin S_{n-1}[i]} 1 \\
&= \sharp|S_{n+1}[j] \setminus S_{n-1}[i]| \\
&= n + 2 - n = 2 = 0 \mod 2
\end{aligned}
$$

Sketch of the proof

- $S_{n-1}[i] \not\subset S_{n+1}[j]$
  $\forall x \in S_{n-1}[i], F(S_{n-1}[i], S_{n+1}[j] \setminus \{x\}) = 0$
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□

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SSReflect

- SSReflect:
  - Extension of Coq
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  - Provides new libraries:

SSReflect

- SSReflect:
  - Extension of Coq
  - Developed while formalizing the Four Color Theorem
  - Provides new libraries:
    - matrix.v: matrix theory
    - finset.v and fintype.v: finite set theory and finite types
    - bigop.v: indexed "big" operations, like $\sum_{i=0}^{n} f(i)$ or $\bigcup_{i \in I} f(i)$
    - zmodp.v: additive group and ring $\mathbb{Z}_p$

Representation of Simplicial Complexes in SSReflect

Definition

Let V be a finite ordered set, called the vertex set, a simplex over V is any finite subset of V

```
Variable V : finType.
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A finite ordered (abstract) simplicial complex over V is a finite set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \textit{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

```
Variable V : finType.
Definition simplex := {set V}.
Definition simplicial_complex (c : {set simplex}) :=
  forall x, x \in c -> forall y : simplex, y \subset x -> y \in c.
```

Incidence Matrices

Definition

Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix $m \times n$ where

$$m = \sharp|X| \ \wedge \ n = \sharp|Y|$$

$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

$$a_{i,j} = \begin{cases} 1 & \text{if } X[i] \text{ is a face of } Y[j] \\ 0 & \text{if } X[i] \text{ is not a face of } Y[j] \end{cases}$$

```
Definition face_op (S : simplex) (x : V) := S :\ x.
Definition boundary (S : simplex) := (face_op S) @: S.

Variables Left Top : {set simplex}.
Definition incidenceMatrix :=
  \matrix_(i < #|Left|, j < #|Top|)
      if enum_val i \in boundary (enum_val j) then 1 else 0:'F_2.
```

Incidence Matrices

Definition

Let C be a finite set of simplices, A be the set of n-simplices of C with an order between its elements and B the set of $(n-1)$-simplices of C with an order between its elements.
We call incidence matrix of dimension n ($n \geq 1$), to a matrix $p \times q$ where

$$p = \sharp|B| \;\wedge\; q = \sharp|A|$$

$$M_{i,j} \quad = \quad \begin{cases} 1 & \textit{if } B[i] \textit{ is a face of } A[j] \\ 0 & \textit{if } B[i] \textit{ is not a face of } A[j] \end{cases}$$

```
Section nth_incidence_matrix.
Variable c: {set simplex}.
Variable n:nat.
Definition n_1_simplices := [set x \in c | #|x| == n].
Definition n_simplices := [set x \in c | #|x| == n+1].
Definition incidence_matrix_n :=
  incidenceMatrix n_1_simplices n_simplices.
End nth_incidence_matrix.
```

Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Theorem (Product of two consecutive incidence matrices in $\mathbb{Z}_2$)

Let $\mathcal{K}$ be a finite simplicial complex over V with an order between the simplices of the same dimension and let $n \geq 1$ be a natural number n, then the product of the n-th incidence matrix of $\mathcal{K}$ and the $(n+1)$-incidence matrix of $\mathcal{K}$ over the ring $\mathbb{Z}/2\mathbb{Z}$ is equal to the null matrix

```
Theorem incidence_matrices_sc_product:
  forall (V:finType) (n:nat) (sc: {set (simplex V)}),
    simplicial_complex sc ->
      (incidence_mx_n sc n) *m (incidence_mx_n sc (n.+1)) = 0.
```

Formalization in SSREFLECT of the theorem

- Summation part:

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  - Lemmas from "bigop" library
  - bigID: $\sum_{i \in r | P_i} F_i = \sum_{i \in r | P_i \wedge a_i} F_i + \sum_{i \in r | P_i \wedge \sim a_i} F_i$
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- Cardinality part:
  - Auxiliary lemmas
  - Lemmas from "finset" and "fintype" libraries

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Conclusions and Further work

- Conclusions:
  - Formalization in Coq/SSReflect:
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Conclusions and Further work

- Conclusions:
  - Formalization in Coq/SSReflect:
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- Further work:
  - Formalization:
    - From digital images to simplicial complexes
    - Computation Smith Normal Form
  - $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$
  - Executability of the proofs:
    - Code extraction
    - Internal computations

Incidence Simplicial Matrices Formalized in Coq/SSReflect*

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CICM 2011, Calculemus track,
July 22, 2011

* Partially supported by Ministerio de Educación y Ciencia, project MTM2009-13842-C02-01, and by European Commission FP7, STREP project ForMath, n. 243847