Proving with ACL2 the correctness of simplicial sets in the Kenzo system[1]

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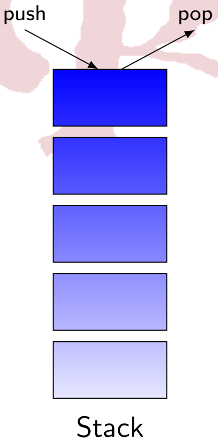
Introductory Example



- Implementation of stacks

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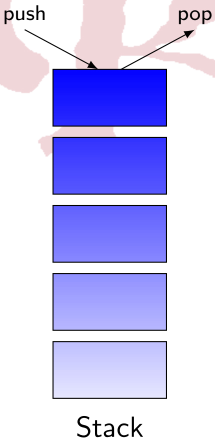

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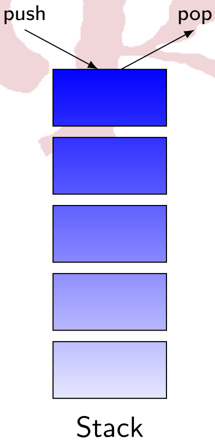

- Implementation of stacks
- Prove the correctness of our implementation
  - Model the problem

```
(defun stack-p (stack)
  (consp stack))

(defun push (elem stack)
  (cons elem stack))

(defun pop (stack)
  (cdr stack))
```

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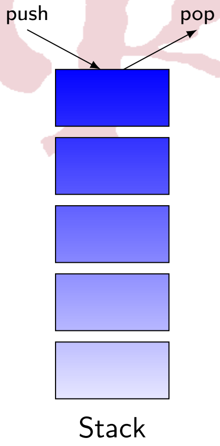

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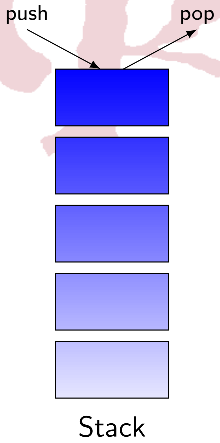

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$\Rightarrow$ Our implementation of a stack is really a stack

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Case Study

Each Kenzo Simplicial Set is really a simplicial set

Table of Contents

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Definition

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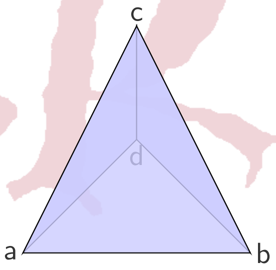

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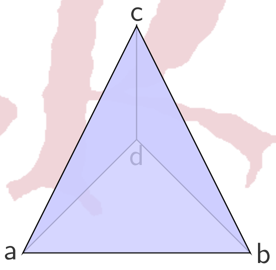

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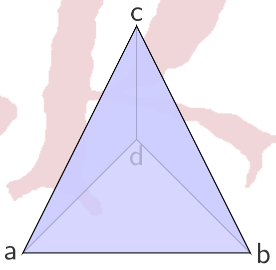

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Proposition

Let K be a simplicial set. Any n-simplex $x \in K^n$ can be expressed in a unique way as a (possibly) iterated degeneracy of a non-degenerate simplex y in the following way:

$$x = \eta_{j_k} \dots \eta_{j_1} y$$

with $y \in K^r$, $k = n - r \geq 0$, and $0 \leq j_1 < \cdots < j_k < n$.

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degenerate	$(a \; a \; b \; c)$	$(\eta_0 \; (a \; b \; c))$

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  - Dependent from the simplicial set ...
  - but some parts are independent
  - $\partial_2(\eta_3\eta_1\ (a\ b\ c)) = (\partial_2\eta_3\eta_1\ (a\ b\ c)) \underset{\partial_{i+1}\eta_i=identity}{\overset{\partial_i\eta_j=\eta_{j-1}\partial_i \text{ if } i<j}{=}} (\eta_2\ (a\ b\ c))$
  - $\partial_2(\eta_3\eta_0\ (a\ b\ c)) = (\partial_2\eta_3\eta_0\ \partial_1(a\ b\ c)) \underset{\partial_i\eta_j = \eta_j\partial_{i-1} \text{ if } i>j+1}{\overset{\partial_i\eta_j=\eta_{j-1}\partial_i \text{ if } i<j}{=}} (\eta_2\eta_0\ (a\ c))$

Mathematical context: minimal conditions

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ *such that for all element* $gmsm \in K^q$ *the following properties hold:*

1. $\forall i, j \in \mathbb{N} : i < j \leq q \Longrightarrow \widehat{\partial}_i^{q-1}(\widehat{\partial}_j^q gmsm) = \widehat{\partial}_{j-1}^{q-1}(\widehat{\partial}_i^q gmsm)$,
2. $\forall i \in \mathbb{N}$, $i \leq q$: $\widehat{\partial}_i^q gmsm \in K^{q-1}$,

then $\{K^q, \partial^q, \eta^q\}_{q\geq 0}$ *is a simplicial set*

ACL2 framework: minimal conditions

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ *such that for all element* $gmsm \in K^q$ *the following properties hold:*

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ACL2 framework: minimal conditions

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Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ *such that for all element gmsm* $\in K^q$ *the following properties hold:*

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then $\{K^q, \partial^q, \eta^q\}_{q\geq 0}$ *is a simplicial set*

```
(encapsulate
 ; Signatures
 (((face * * *)  => *)
  ((dimension *) => *)
  ((canonical *) => *)
  ((inv-ss * *)  => *))
 ...
 )
```

ACL2 framework: minimal conditions

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ *such that for all element gmsm* $\in K^q$ *the following properties hold:*

1. $\forall i, j \in \mathbb{N} : i < j \leq q \Longrightarrow \widehat{\partial}_i^{q-1}(\widehat{\partial}_j^q gmsm) = \widehat{\partial}_{j-1}^{q-1}(\widehat{\partial}_i^q gmsm)$,
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```
(encapsulate
 ; Signatures
 (((face * * *)  => *)
  ((dimension *) => *)
  ((canonical *) => *)
  ((inv-ss * *)  => *))
; Theorems
 (defthm faceoface
   (implies (and (natp i) (natp j) (< i j) (inv-ss ss ls))
    (equal (face ss i (face ss j ls)) (face ss (- j 1) (face ss i ls)))))
```

ACL2 framework: minimal conditions

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ *such that for all element gmsm* $\in K^q$ *the following properties hold:*

1. $\forall i,j \in \mathbb{N} : i < j \leq q \Longrightarrow \widehat{\partial}_i^{q-1}(\widehat{\partial}_j^q gmsm) = \widehat{\partial}_{j-1}^{q-1}(\widehat{\partial}_i^q gmsm)$,
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```
(encapsulate
 ; Signatures
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 (defthm faceoface
   (implies (and (natp i) (natp j) (< i j) (inv-ss ss ls))
    (equal (face ss i (face ss j ls)) (face ss (- j 1) (face ss i ls)))))
(defthm inv-ss-prop
  (implies (and (canonical absm) (natp i) (< i (dimension absm)))
  (equal (dimension (face ss i absm)) (1- (dimension absm)))
; Witness ... )
```

ACL2 framework: face and degeneracy

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ such that for all element gmsm $\in K^q$ the following properties hold:

1. $\forall i, j \in \mathbb{N} : i < j \leq q \Longrightarrow \widehat{\partial}_i^{q-1}(\widehat{\partial}_j^q gmsm) = \widehat{\partial}_{j-1}^{q-1}(\widehat{\partial}_i^q gmsm)$,
2. $\forall i \in \mathbb{N}$, $i \leq q$: $\partial_i^q gmsm \in K^{q-1}$,

then $\{K^q, \partial^q, \eta^q\}_{q\geq 0}$ is a simplicial set

```
(defun imp-face-Kenzo (ss i q (dgop gmsm))
    (if (face-absm-dgop i dgop)
        (list (face-absm-dgop i dgop) gmsm)
      (list (face-absm-dgop i dgop) (face ss (face-absm-indx i dgop) gmsm)))))

(defun imp-degeneracy-Kenzo (ss i q (dgop gmsm))
  (list (degeneracy-absm-dgop-dgop i dgop) gmsm))

(defun imp-inv-Kenzo (ss q (dgop gmsm))
  ...)
```

`imp-inv-Kenzo` is the characteristic function

ACL2 framework: Proof of Theorem

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q \geq 0}$ *such that for all element gmsm* $\in K^q$ *the following properties hold:*

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then $\{K^q, \partial^q, \eta^q\}_{q \geq 0}$ *is a simplicial set*

- `imp-face-Kenzo` and `imp-degeneracy-Kenzo` are well-defined

```
(defthm theorem-1
(implies (imp-inv-Kenzo ss q (dgop gmsm))
         (imp-inv-Kenzo ss (1- q) (imp-face-Kenzo ss i q (dgop gmsm)))))
```

ACL2 framework: Proof of Theorem

Theorem

Let the object $\{K^q, \widehat{\partial}^q\}_{q\geq 0}$ such that for all element gmsm $\in K^q$ the following properties hold:

1. $\forall i, j \in \mathbb{N} : i < j \leq q \Longrightarrow \partial_i^{q-1}(\partial_j^q gmsm) = \partial_{j-1}^{q-1}(\partial_i^q gmsm)$,
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then $\{K^q, \partial^q, \eta^q\}_{q\geq 0}$ is a simplicial set

- imp-face-Kenzo and imp-degeneracy-Kenzo are well-defined

```
(defthm theorem-1
(implies (imp-inv-Kenzo ss q (dgop gmsm))
         (imp-inv-Kenzo ss (1- q) (imp-face-Kenzo ss i q (dgop gmsm)))))
```

- imp-face-Kenzo and imp-degeneracy-Kenzo satisfy the 5 properties of simplicial sets

```
(defthm theorem-3
(implies (and (imp-inv-Kenzo ss q (dgop gmsm)) (natp i) (natp j) (< i j))
         (equal (imp-face-Kenzo ss i (1- q) (imp-face-Kenzo ss j q (dgop gmsm)))
                (imp-face-Kenzo ss (1- j) (1- q) (imp-face-Kenzo ss i q (dgop gmsm))))))
```

Sketch of the proofs

Methodological approach imported from:

F. J. Martín-Mateos, J. Rubio, and J. L. Ruiz-Reina. ACL2 verification of simplical degeneracy programs in the Kenzo system. Lecture Notes in Computer Science, 5625:106–121, 2009.

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1. Prove each theorem with EAT representation

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   - EAT is the predecessor of Kenzo

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    - EAT is the predecessor of Kenzo
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   - EAT is the predecessor of Kenzo
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   - Closer to mathematical representation
2. Prove the equivalence between Kenzo and EAT functions module a domain transformation

`imp-face-eat`	$\Leftrightarrow$	`imp-face-Kenzo`
`imp-degeneracy-eat`	$\Leftrightarrow$	`imp-degeneracy-Kenzo`
`imp-inv-eat`	$\Leftrightarrow$	`imp-inv-Kenzo`

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`imp-inv-eat`	$\Leftrightarrow$	`imp-inv-Kenzo`

$\Rightarrow$ All the theorems are proved with Kenzo representation

EAT/Kenzo representation

EAT

Kenzo

EAT/Kenzo representation

EAT

- abstract simplexes:

  (*dgop* *gmsm*) :=

  $\begin{cases} \textit{dgop} \text{ is a strictly decreasing list} \\ \textit{gmsm} \text{ is an object} \end{cases}$

  Example:

  $(\eta_3\eta_1\ (a\ b\ c)) \rightsquigarrow ((3\ 1)\ (a\ b\ c))$

Kenzo

- abstract simplexes:

  (*dgop* *gmsm*) :=

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  Example:
  $(\eta_3\eta_1\ (a\ b\ c)) \rightsquigarrow (10\ (a\ b\ c))$

  $\eta_3\eta_1 \rightsquigarrow (0\ 1\ 0\ 1) \rightsquigarrow$

  $0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 10$

EAT/Kenzo representation

EAT

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- face, degeneracy:
  implemented with recursive functions

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  Example:
  $(\eta_3\eta_1\ (a\ b\ c)) \rightsquigarrow (10\ (a\ b\ c))$

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  $0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = 10$

- face, degeneracy:
  implemented using efficient primitives dealing with binary numbers

EAT/Kenzo representation

EAT

- abstract simplexes:

  $(dgop \;\; gmsm) :=$

  $\begin{cases} dgop \text{ is a strictly decreasing list} \\ gmsm \text{ is an object} \end{cases}$

  Example:

  $(\eta_3\eta_1 \; (a \; b \; c)) \rightsquigarrow ((3\;1) \; (a \; b \; c))$

- face, degeneracy:
  implemented with recursive functions

- inefficient

- easy to prove

Kenzo

- abstract simplexes:

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- face, degeneracy:
  implemented using efficient primitives
  dealing with binary numbers

- efficient

- difficult to prove

Proof of a theorem

- We want to prove

```
(defthm theorem-3-Kenzo
(implies (and (imp-inv-Kenzo ss q (dgop gmsm)) (natp i) (natp j) (< i j))
        (equal (imp-face-Kenzo ss i (1- q) (imp-face-Kenzo ss j q (dgop gmsm)))
               (imp-face-Kenzo ss (1- j) (1- q) (imp-face-Kenzo ss i q (dgop gmsm))))))
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                (imp-face-Kenzo ss (1- j) (1- q) (imp-face-Kenzo ss i q (dgop gmsm))))))
```

1. First we prove

```
(defthm theorem-3-eat
(implies (and (imp-inv-eat ss q (dgop gmsm)) (natp i) (natp j) (< i j))
         (equal (imp-face-eat ss i (1- q) (imp-face-eat ss j q (dgop gmsm)))
                (imp-face-eat ss (1- j) (1- q) (imp-face-eat ss i q (dgop gmsm))))))
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- We want to prove

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                (imp-face-eat ss (1- j) (1- q) (imp-face-eat ss i q (dgop gmsm))))))
```

   - induction
   - simplification
   - study of cases

Proof of a theorem continued

2. then we prove

   `imp-face-eat` $\Leftrightarrow$ `imp-face-Kenzo`

Proof of a theorem continued

2. then we prove

   `imp-face-eat` $\Leftrightarrow$ `imp-face-Kenzo`

- Difficult to prove
  - Kenzo and EAT deal with different representations
  - Kenzo implementation is not intuitive

Proof of a theorem continued

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mathematical	EAT	Binary	Kenzo
$\eta_3\eta_1$	(3 1)	(0 1 0 1)	10

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- Definition of `imp-face-binary`
  - Works with binary lists
  - Inspired from Kenzo functions

Proof of a theorem continued

2. then we prove

`imp-face-eat` $\Leftrightarrow$ `imp-face-Kenzo`

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`imp-face-eat` $\Leftrightarrow$ `imp-face-binary` $\Leftrightarrow$ `imp-face-Kenzo`

Distance from ACL2 code to actual Kenzo code: values

Kenzo

```
(defun 1dlop-dgop (1dlop dgop)
  (progn
    (when (logbitp 1dlop dgop)
      (let ((share (ash -1 1dlop)))
        (values
         (logxor
          (logand share (ash dgop -1))
          (logandc1 share dgop))
         nil)))
    (when (and (plusp 1dlop)
               (logbitp (1- 1dlop) dgop))
      (let ((share (ash -1 1dlop)))
        (setf share (ash share -1))
        (return-from 1dlop-dgop
          (values
           (logxor
            (logand share (ash dgop -1))
            (logandc1 share dgop))
           nil))))
    (let ((share (ash -1 1dlop)))
      (let ((right (logandc1 share dgop)))
        (values
         (logxor
          right
          (logand share (ash dgop -1)))
         (- 1dlop (logcount right)))))))
```

ACL2

```
(defun 1dlop-dgop-dgop (1dlop dgop)
  (if (and (natp 1dlop) (natp dgop))
      (cond ((logbitp 1dlop dgop)
             (logxor
              (logand (ash -1 1dlop)
                      (ash dgop -1))
              (logandc1 (ash -1 1dlop)
                        dgop)))
            ((and (plusp 1dlop)
                  (logbitp (- 1dlop 1) dgop))
             (logxor
              (logand (ash (ash -1 1dlop) -1)
                      (ash dgop -1))
              (logandc1 (ash (ash -1 1dlop) -1)
                        dgop)))
            (t (logxor
                (logandc1 (ash -1 1dlop) dgop)
                (logand (ash -1 1dlop)
                        (ash dgop -1)))))
    nil))

(defun 1dlop-dgop-indx (1dlop dgop)
  (if (or (logbitp 1dlop dgop)
          (and (plusp 1dlop)
               (logbitp (- 1dlop 1) dgop)))
      nil
    (- 1dlop
       (logcount (logandc1 (ash -1 1dlop) dgop)
```

Distance from ACL2 code to actual Kenzo code: values

Kenzo

```
(defun 1dlop-dgop (1dlop dgop)
  (progn
    (when (logbitp 1dlop dgop)
      (let ((share (ash -1 1dlop)))
        (values
         (logxor
          (logand share (ash dgop -1))
          (logandc1 share dgop))
        nil)))
    (when (and (plusp 1dlop)
               (logbitp (1- 1dlop) dgop))
      (let ((share (ash -1 1dlop)))
        (setf share (ash share -1))
        (return-from 1dlop-dgop
          (values
           (logxor
            (logand share (ash dgop -1))
            (logandc1 share dgop))
           nil))))
    (let ((share (ash -1 1dlop)))
      (let ((right (logandc1 share dgop)))
        (values
         (logxor
          right
          (logand share (ash dgop -1)))
         (- 1dlop (logcount right)))))))
```

ACL2

```
(defun 1dlop-dgop-dgop (1dlop dgop)
  (if (and (natp 1dlop) (natp dgop))
      (cond ((logbitp 1dlop dgop)
             (logxor
              (logand (ash -1 1dlop)
                      (ash dgop -1))
              (logandc1 (ash -1 1dlop)
                        dgop)))
            ((and (plusp 1dlop)
                  (logbitp (- 1dlop 1) dgop))
             (logxor
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  (if (or (logbitp 1dlop dgop)
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      nil
    (- 1dlop
       (logcount (logandc1 (ash -1 1dlop) dgop)
```

Distance from ACL2 code to actual Kenzo code: loops

Kenzo

```
(defun cmp-d-ls-dgop (d ls)
  (do ((p ls (cdr p))
       (rsl
         empty-list (let ((j (car p)))
                      (cons (cond ((< d j) (1- j))
                                  (t (decf d) j))
                                    rsl))))
      ((endp p) (nreverse rsl))
    (when (<= 0 (- d (car p)) 1)
      (return (nreconc rsl (rest p))))))
```

ACL2

```
(defun cmp-d-ls-dgop-do (d p rsl)
  (cond ((endp p) (reverse rsl))
        ((< d (car p))
         (cmp-d-ls-dgop-do d (cdr p)
                     (cons (1- (car p)) rsl)))
        ((and (<= 0 (- d (car p)))
              (<= (- d (car p)) 1))
         (append (reverse rsl) (rest p)))
        (t (cmp-d-ls-dgop-do (1- d)
                  (cdr p) (cons (car p) rsl)))
        )
  )

(defun cmp-d-ls-dgop (d ls)
  (cmp-d-ls-dgop-do d ls nil)
  )
```

Distance from ACL2 code to actual Kenzo code: loops

Kenzo

```
(defun cmp-d-ls-dgop (d ls)
  (do ((p ls (cdr p))
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         (cmp-d-ls-dgop-do d (cdr p)
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        ((and (<= 0 (- d (car p)))
              (<= (- d (car p)) 1))
         (append (reverse rsl) (rest p)))
        (t (cmp-d-ls-dgop-do (1- d)
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        )
  )

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  (cmp-d-ls-dgop-do d ls nil)
  )
```

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    - Instantiates definitions and theorems of the theory for different instances (different simplicial sets)

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(defun face-delta (n i gmsm)
   (cond ((zp i) (cdr gmsm))
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  2. Proof of the four theorems:

```
(defthm faceoface-delta
 (implies (and (natp i) (natp j) (< i j) (canonical-delta gmsm))
          (equal (face-delta n i (face-delta n j gmsm))
                 (face-delta n (+ -1 j) (face-delta n i gmsm)))))
...
```

Certifications of Simplicial Set families

3. Instantiation of the theory:

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(definstance-*simplicial-set-kenzo*
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4. A proof of Kenzo Standard Simplicial Sets are really Simplicial Sets is automatically generated

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Proving with ACL2 the correctness of simplicial sets in the Kenzo system

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