Coherent and Strongly Discrete Rings in Type Theory

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Introduction

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

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Introduction

```
A=[ 0 2 ; 1 0 ];
b=[ 2 ; 0 ];
A'\b
```

Introduction

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A=[ 0 2 ; 1 0 ];
b=[ 2 ; 0 ];
A'\b

ans =
0
1
```

Introduction

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Introduction

- *"The bug seems to be new in release R2009b and applies to Linux and Windows 32 and 64 bit versions."*
  `http://www.netlib.org/na-digest-html/09/v09n48.html`

- *A serious bug in Matlab2009b?*
  `http://www.walkingrandomly.com/?p=1964`

Introduction

Goal: Formally verified algorithms for solving (in)homogeneous systems of equations over *commutative rings* in Coq/SSReflect

Motivation: Simplifying systems of differential equations, computing homology groups...

Coherent rings

For every row matrix $M \in R^{1\times m}$ there exists $L \in R^{m\times n}$ such that

$$MX = 0 \leftrightarrow \exists Y . X = LY$$

L generate the module of solutions for $MX = 0$

Coherent rings

Theorem: Possible to solve $MX = 0$ where $M \in R^{m \times n}$, i.e. we can compute L such that

$$MX = 0 \leftrightarrow \exists Y . X = LY$$

Constructive proof $\Rightarrow$ algorithm computing generators of solutions

Coherent rings

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Constructive proof $\Rightarrow$ algorithm computing generators of solutions

Constructive proof $=$ Program $+$ Specification $+$ Correctness proof

Coherent rings

```
Fixpoint solve m n : forall (M : 'M_(m,n)),
  'M_(n,size_solve M) := match m with
    | S p => fun (M : 'M_(1+p,n)) =>
      let G1 := solve_row (usubmx M) in
      G1 *m solve (dsubmx M *m G1)
    | _ => fun _ => 1%:M
  end.

Lemma solveP m n (M : 'M[R]_(m,n)) (X : 'cV[R]_n) :
 reflect (exists Y, X = solve M *m Y) (M *m X == 0).
Proof.
...
Qed.
```

Coherent rings

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Problems: How do we represent $I \cap J$?

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Theorem: If $I \cap J$ is finitely generated for finitely generated ideals I and J in R then R is coherent.

Problems: How do we represent $I \cap J$? How do we represent ideals?

Strongly discrete rings

There exists an algorithm testing if $x \in I$ for finitely generated ideal I and if this is the case produce a witness.

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That is, if $I = (x_1, \dots, x_n)$ compute $w_1, \dots, w_n$ such that

$$x = x_1 w_1 + \cdots + x_n w_n$$

Strongly discrete rings

- Suitable for developing ideal theory in type theory:
  - Decidable ideal membership: $x \in I$
  - Decidable ideal inclusion: $I \subseteq J$
  - Decidable ideal equality: $I = J \leftrightarrow I \subseteq J \wedge J \subseteq I$

- Key idea: Represent *finitely generated* ideals as row matrices

Ideals

```
Definition subid (I : 'rV[R]_m) (J : 'rV[R]_n) := ...

Notation "I <= J" := (subid I J).

Lemma subidP (I : 'rV[R]_m) (J : 'rV[R]_n) :
  reflect (exists D, I = J *m D) (I <= J)%IS.

Definition addid (I : 'rV[R]_m) (J : 'rV[R]_n) :=
  row_mx I J.

Notation "I +i J" := (addid I J).

Lemma subid_addidC (I : 'rV[R]_m) (J : 'rV[R]_n) :
  (I +i J <= J +i I)%IS.
```

$I \cap J$ revisited

- $I \cap J$ can be defined as an ideal such that:

$$I \cap J \subseteq I$$

$$I \cap J \subseteq J$$

$$\forall x.\ (x \in I \wedge x \in J) \rightarrow x \in I \cap J$$

- Now we can prove (*constructively*) that if $I \cap J$ is finitely generated then R is coherent

Coherent strongly discrete rings

Recap:

- R is *coherent* if we can find generators for solutions of $MX = 0$
- R is *strongly discrete* if we can decide if $x \in I$ for a finitely generated ideal I in R

Theorem: If R is coherent and strongly discrete we can decide if a system $MX = A$ has a solution

Coherent rings

Examples of coherent rings:

- Fields – Gaussian elimination
- Bézout domains – $\mathbb{Z}$, $\mathbb{Q}[x]$, ...
- Prüfer domains – $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Q}[x,y]/(y^2-1+x^4)$, ...
- Polynomial rings – $k[x_1, \ldots, x_n]$ via Gröbner bases
- ...

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Bézout domains

- Bézout domains: Every *finitely generated* ideal is principal, i.e. for all finitely generated ideals I there is $a \in R$ such that $I = (a)$
- Equivalent definition:

$$\forall ab. \exists xy.\ ax + by = gcd(a, b)$$

Bézout domains

Theorem: Bézout domains (with explicit divisibility) are strongly discrete

Theorem: Bézout domains are coherent

Get algorithm for solving $MX = A$ over $\mathbb{Z}$ and $k[x]$

Prüfer domains

- First-order characterization:

$$\forall ab. \exists uvw.\ ua = vb \wedge (1-u)b = wa$$

- Has many nice ideal properties
- Examples: Bézout domains, algebraic numbers ($\mathbb{Z}[\sqrt{-5}]$), algebraic curves ($\mathbb{Q}[x,y]/(y^2-1+x^4)$), ...

Prüfer domains

Theorem: Prüfer domains (with explicit divisibility) are strongly discrete

Theorem: Prüfer domains are coherent

Get algorithm for solving system $MX = A$ over Prüfer domains

Computation in Coq

- Matrices in SSReflect are represented as:

```
Inductive matrix R m n :=
  Matrix of {ffun 'I_m * 'I_n -> R}.
```

- Well suited for proofs, but not for computation...

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- Solution: Data refinements

Data refinements

```
'M[R]_(m,n)
```

```
seq (seq R)
```

Data refinements

$$\begin{tikzcd}
\texttt{'M[R]_(m,n)} \arrow[d, "\texttt{seqmx_of_mx}"] \\
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Data refinements

$$\begin{tikzcd}
\begin{array}{c} \texttt{'M[R]_(m,n)} \\ +,\ \times,\ \cdot^{T},\ldots \end{array} \arrow[d, "\texttt{seqmx_of_mx}"] \\
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Computations

Solve

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```
Eval compute in
  csolveGeneral 2 2
    (trseqmx [::[:: 0; 2];[:: 1; 0]])
     [::[:: 2];[:: 0]].
  = Some [:: [:: 0]; [:: 2]]
```

Summary and conclusions

- Implementation of formally verified algorithms for solving systems of equations over:
  - Bézout domains: $\mathbb{Z}$, $\mathbb{Q}[x]$, ...
  - Prüfer domains: $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Q}[x,y]/(y^2 - 1 + x^4)$,...
- Using data refinements to implement efficient algorithms works well
- CoqEAL[1] – The Coq Effective Algebra Library

[1] http://www-sop.inria.fr/members/Maxime.Denes/coqeal/

Thank you!

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Extra slides

Bézout domains: Coherent

Theorem: Bézout domains are coherent

- It suffices to show that $I \cap J$ is finitely generated for finitely generated ideals I and J.
- Compute a and b such that $I = (a)$ and $J = (b)$.
- Can prove $I \cap J = (lcm(a, b))$

Prüfer domains

Theorem: Every nonzero finitely generated ideal is *invertible*

Given a finitely generated ideal I over R there exists I^{-1} such that $II^{-1} = (a)$ for some $a \in R$

Coherence of Prüfer domains

Theorem: Prüfer domains are coherent

- Given finitely generated I and J we have $(I + J)(I \cap J) = IJ$
- To compute $I \cap J$ invert $I + J$ and multiply on both sides:

$$(a)(I \cap J) = (I + J)^{-1} IJ$$

- Hence we can compute $I \cap J$

Optimizations

- We can also do program refinements to optimize our algorithms
- Simple example: Ideal addition
  - Problems with naive implementation: Zeroes? Duplicate elements?
  - Solution: Implement more efficient version $+'$ such that $I + J = I +' J$ and then refine $+'$ to lists

Future work

- Polynomial rings – $k[x_1, \ldots, x_n]$ via Gröbner bases
- Implement library of homological algebra – HOMALG project[2]

[2] `http://homalg.math.rwth-aachen.de/`

SSReflect matrices

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Inductive matrix R m n :=
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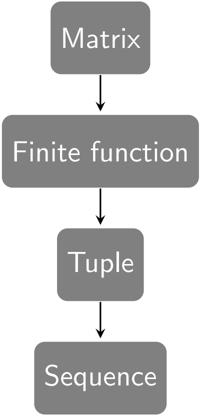

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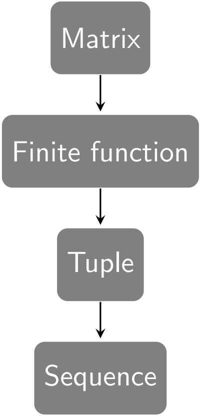


- Fine-grained architecture
- Proof-oriented design
- Had to be locked to avoid term size explosion
- Not suited for computation

Objective

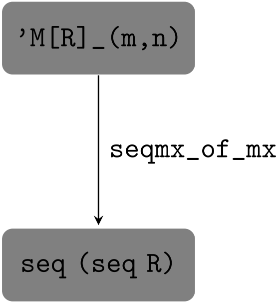
- Define concrete and executable representations and operations on matrices, using a relaxed datatype (unsized lists)
- Devise a way to link them with the theory in SSReflect
- Still be able to use convenient tools to reason about the algorithms we implement

Methodology

`'M[R]_(m,n)`

`seq (seq R)`

Methodology



Methodology

'M[R]_(m,n)

$+, \times, .^T, \ldots$

$\big\downarrow$ seqmx_of_mx

seq (seq R)

addseqmx,

mulseqmx,

trseqmx, ...

List-based representation of matrices

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  reflect (M = N) (seqmx_of_mx M == seqmx_of_mx N).

Definition addseqmx (M N : seqmatrix) : seqmatrix :=
  zipwith (zipwith (fun x y => x + y)) M N.

Lemma addseqmxE (M N : 'M[R]_(m,n)) :
  seqmx_of_mx (M + N) =
    addseqmx (seqmx_of_mx M) (seqmx_of_mx N).
```

Methodology

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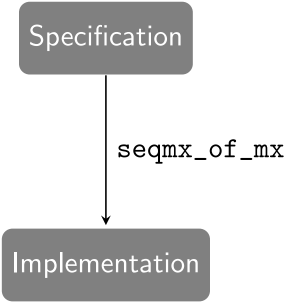
Methodology

Specification

↓ `seqmx_of_mx`

`seq (seq R)`
`addseqmx,`
`mulseqmx,`
`trseqmx, ...`

Methodology



Methodology