Formal libraries for Algebraic Topology: status report[1]

ForMath La Rioja node
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Contributors

- Local Contributors:
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  - César Domínguez
  - Jónathan Heras
  - Laureano Lambán
  - Vico Pascual
  - María Poza
  - Julio Rubio

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- Contributors from Universidad de Sevilla:
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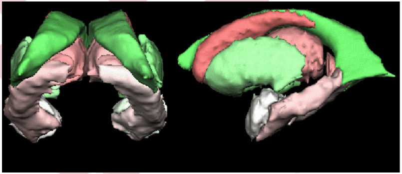
Goal

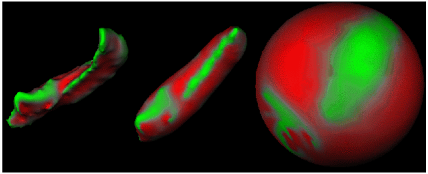
- Formalization of libraries for Algebraic Topology

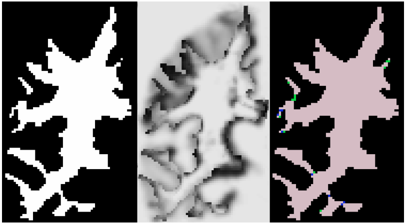
Goal

- Formalization of libraries for Algebraic Topology
  - Application: Study of digital images

Applying topological concepts to analyze images







F. Ségonne, E. Grimson, and B. Fischl. Topological Correction of Subcortical Segmentation. International Conference on Medical Image Computing and Computer Assisted Intervention, MICCAI 2003, LNCS 2879, Part 2, pp. 695-702.

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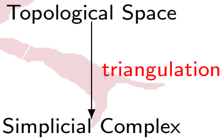
Topological Space

From General Topology to Algebraic Topology



From General Topology to Algebraic Topology

Topological Space

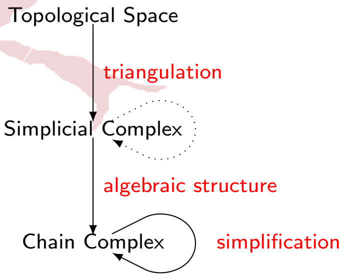
↓ triangulation

Simplicial Complex

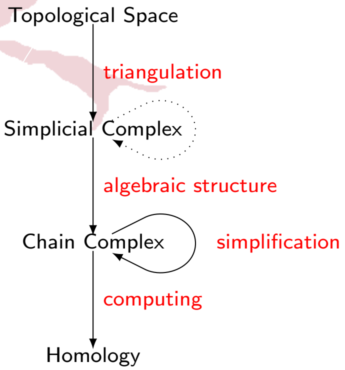
↓ algebraic structure

Chain Complex

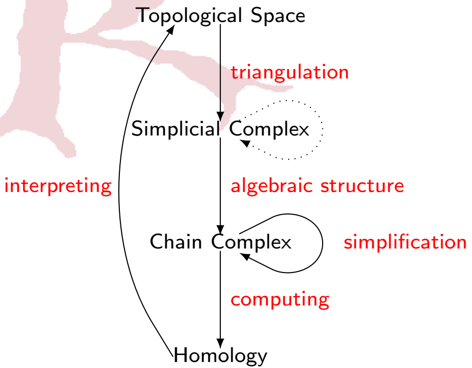
From General Topology to Algebraic Topology



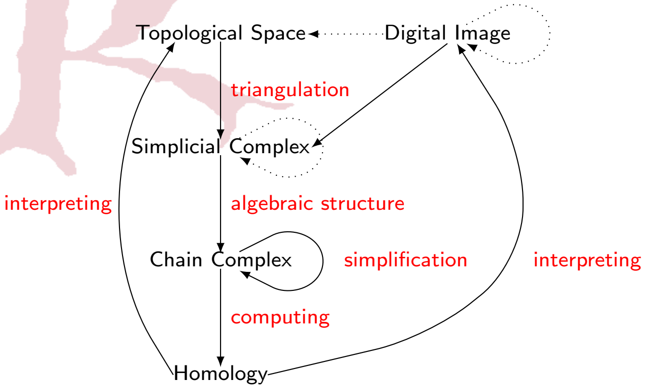
From General Topology to Algebraic Topology



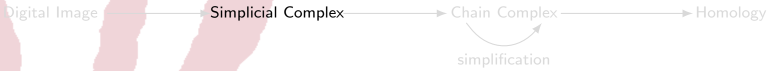
From General Topology to Algebraic Topology



From General Topology to Algebraic Topology



Simplicial Complexes



Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Simplicial Complexes

Digital Image → **Simplicial Complex** → Chain Complex → Homology

(Chain Complex ↻ simplification)

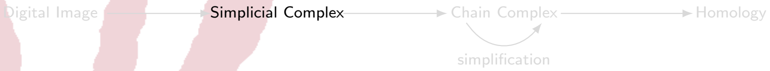
Definition

Let V be an ordered set, called the vertex set.
A simplex over V is any finite subset of V.

Definition

Let α and β be simplices over V, we say α is a face of β if α is a subset of β.

Simplicial Complexes



Definition

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A simplex over V is any finite subset of V.

Definition

Let α and β be simplices over V, we say α is a face of β if α is a subset of β.

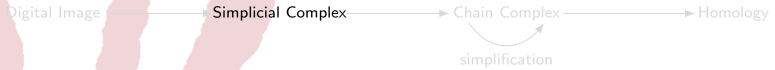
Definition

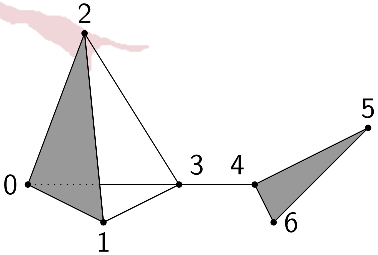
An ordered (abstract) simplicial complex over V is a set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K}, \text{ if } \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

Let $\mathcal{K}$ be a simplicial complex. Then the set $S_n(\mathcal{K})$ of n-simplices of $\mathcal{K}$ is the set made of the simplices of cardinality $n+1$.

Simplicial Complexes





$V = (0, 1, 2, 3, 4, 5, 6)$
$\mathcal{K} = \{\emptyset, (0), (1), (2), (3), (4), (5), (6),$
$(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3), (3, 4), (4, 5), (4, 6), (5, 6),$
$(0, 1, 2), (4, 5, 6)\}$

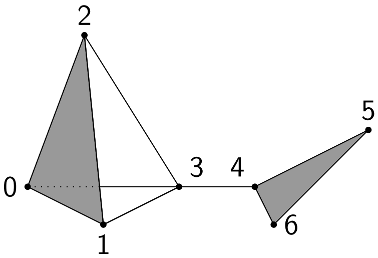
Simplicial Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification

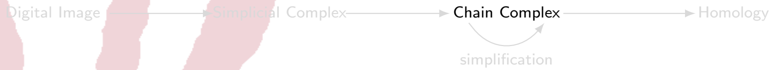
Definition

The facets of a simplicial complex $\mathcal{K}$ are the maximal simplices of the simplicial complex.



The facets are: $\{(0,3),(1,3),(2,3),(3,4),(0,1,2),(4,5,6)\}$

Chain Complexes

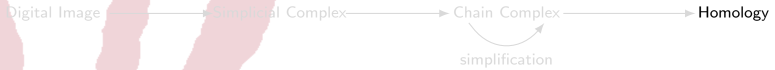


Definition

A chain complex C_* *is a pair of sequences* $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ *where:*

- *For every* $q \in \mathbb{Z}$, *the component* C_q *is an* R*-module, the chain group of degree* q
- *For every* $q \in \mathbb{Z}$, *the component* d_q *is a module morphism* $d_q : C_q \to C_{q-1}$, *the differential map*
- *For every* $q \in \mathbb{Z}$, *the composition* $d_q d_{q+1}$ *is null:* $d_q d_{q+1} = 0$

Homology



Definition

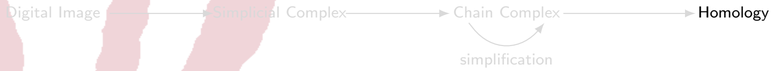
If $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ is a chain complex:

- *The image $B_q = im\ d_{q+1} \subseteq C_q$ is the (sub)module of q-boundaries*
- *The kernel $Z_q = ker\ d_q \subseteq C_q$ is the (sub)module of q-cycles*

Given a chain complex $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$:

- $d_{q-1} \circ d_q = 0 \Rightarrow B_q \subseteq Z_q$
- Every boundary is a cycle
- The converse is not generally true

Homology



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- Every boundary is a cycle
- The converse is not generally true

Definition

Let $C_* = (C_q, d_q)_{q \in \mathbb{Z}}$ be a chain complex. For each degree $n \in \mathbb{Z}$, the n-homology module of C_* is defined as the quotient module

$$H_n(C_*) = \frac{Z_n}{B_n}$$

From Simplicial Complexes to Chain Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification

Definition

Let $\mathcal{K}$ be an (ordered abstract) simplicial complex. Let $n \geq 1$ and $0 \leq i \leq n$ be two integers n and i. Then the face operator ∂_i^n is the linear map $\partial_i^n : S_n(\mathcal{K}) \rightarrow S_{n-1}(\mathcal{K})$ defined by:

$$\partial_i^n((v_0, \dots, v_n)) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n).$$

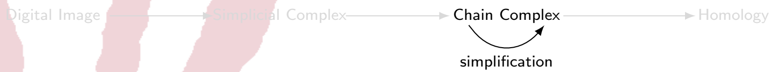
The i-th vertex of the simplex is removed, so that an $(n-1)$-simplex is obtained.

Definition

Let $\mathcal{K}$ be a simplicial complex. Then the chain complex $C_*(\mathcal{K})$ canonically associated with $\mathcal{K}$ is defined as follows. The chain group $C_n(\mathcal{K})$ is the free $\mathbb{Z}$ module generated by the n-simplices of $\mathcal{K}$. In addition, let $(v_0, \dots, v_{n-1})$ be a n-simplex of $\mathcal{K}$, the differential of this simplex is defined as:

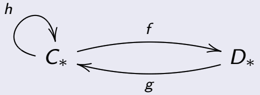
$$d_n := \sum_{i=0}^{n} (-1)^i \partial_i^n$$

Simplification: Perturbation techniques



Definition

A reduction ρ between two chain complexes C_* y D_* (denoted by $\rho : C_* \Rightarrow D_*$) is a triple $\rho = (f, g, h)$



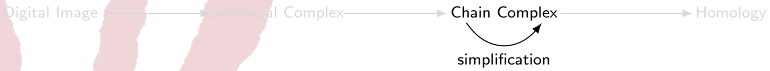
satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

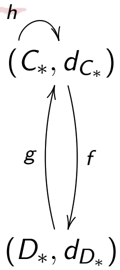
Theorem

If $C_* \Rightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

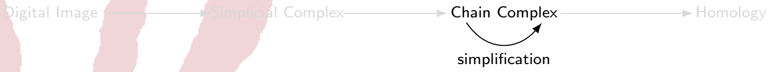
Simplification: Perturbation techniques

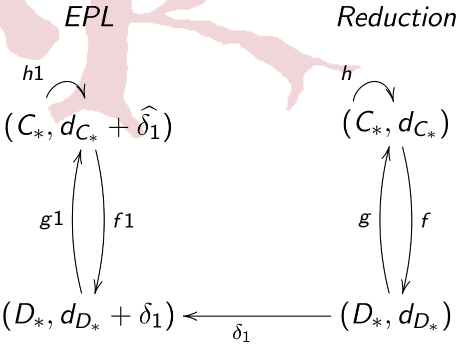


Reduction



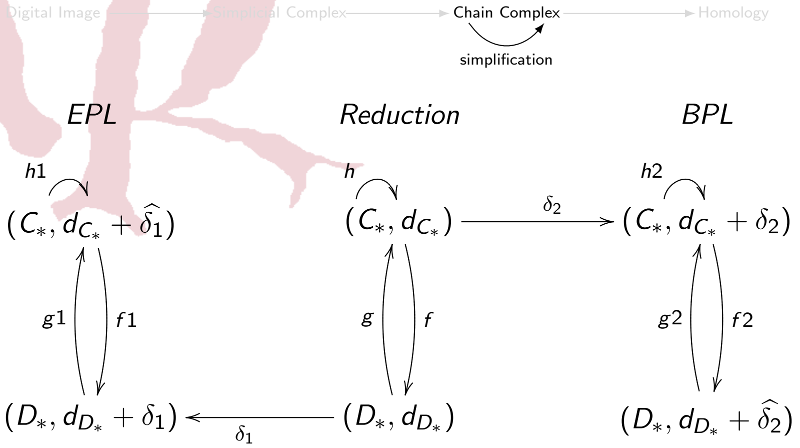
Simplification: Perturbation techniques





Easy Perturbation Lemma (EPL)

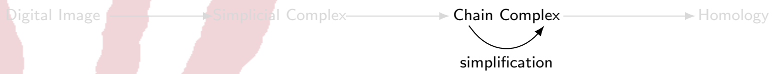
Simplification: Perturbation techniques



Easy Perturbation Lemma (EPL)

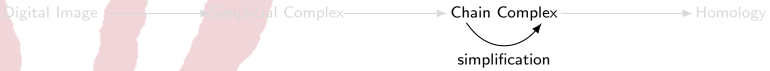
Basic Perturbation Lemma (BPL)

Simplification: Discrete Morse Theory



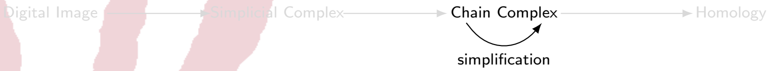
- Discrete Morse Theory:
  - Discrete vector fields *DVF*

Simplification: Discrete Morse Theory



- Discrete Morse Theory:
  - Discrete vector fields *DVF*
  - Critical cells

Simplification: Discrete Morse Theory



- Discrete Morse Theory:
  - Discrete vector fields *DVF*
  - Critical cells
  - From a chain complex C_* and a *DVF* V on C_* constructs a reduction from C_* to C_*^c where generators of C_*^c are the critical cells of C_* with respect to V

Simplification: Discrete Morse Theory

Digital Image → Simplicial Complex → **Chain Complex** → Homology

simplification

- Discrete Morse Theory:
  - Discrete vector fields *DVF*
  - Critical cells
  - From a chain complex C_* and a *DVF* V on C_* constructs a reduction from C_* to C_*^c where generators of C_*^c are the critical cells of C_* with respect to V

R. Forman. Morse theory for cell complexes. Advances in Mathematics, 134:90-145, 1998.

A. Romero, F. Sergeraert. Discrete Vector Fields and Fundamental Algebraic Topology. http://arxiv.org/abs/1005.5685.

Computing

Digital Image → Simplicial Complex → Chain Complex → Homology

(simplification)

- Computing Homology groups:
  - From a Chain Complex $(C_n, d_n)_{n\in\mathbb{Z}}$ of finite type
    - d_n can be expressed as matrices
    - Homology groups are obtained from a diagonalization process

Computing

Digital Image → Simplicial Complex → Chain Complex → Homology

(simplification)

- Computing Homology groups:
  - From a Chain Complex $(C_n, d_n)_{n \in \mathbb{Z}}$ of finite type
    - d_n can be expressed as matrices
    - Homology groups are obtained from a diagonalization process
  - From a Chain Complex $(C_n, d_n)_{n \in \mathbb{Z}}$ of non finite type
    - Effective Homology Theory
    - Reductions

J. Rubio and F. Sergeraert. Constructive Algebraic Topology, Bulletin des Sciences Mathématiques, 126:389-412, 2002.

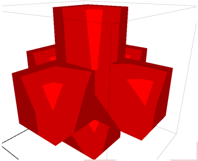
Digital Images

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification

- 2D digital images:
  - elements are pixels

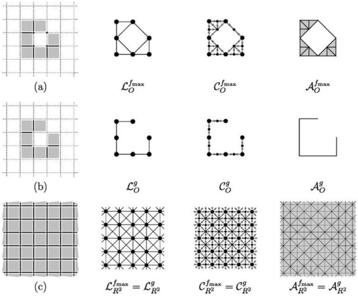
- 3D digital images:
  - elements are voxels



From a digital image to simplicial complexES

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification



R. Ayala, E. Domínguez, A.R. Francés, A. Quintero. Homotopy in digital spaces. Discrete Applied Mathematics 125 (2003) 3-24.

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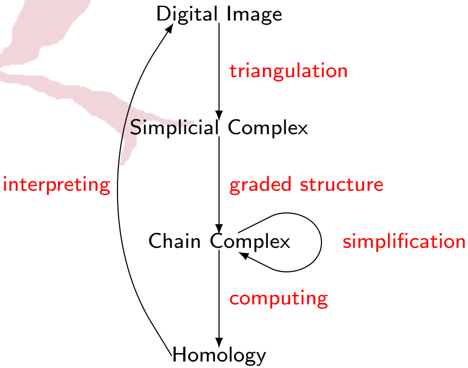
Computing in Algebraic Topology



Demonstration *fKenzo*: user interface for Sergeraert's Kenzo system

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Formalization of Simplicial Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification

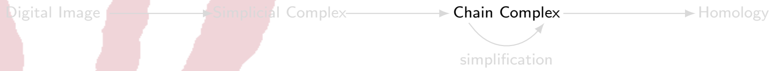
- Formalized in ACL2

J. Heras, V. Pascual and J. Rubio. ACL2 verification of Simplicial Complexes programs for the Kenzo system. Preprint.

- Formalization in Coq/SSReflect

Y. Bertot, L. Rideau and ForMath La Rioja node. Technical report on a SSReflect week. http://wiki.portal.chalmers.se/cse/pmwiki.php/ForMath/ForMath.

Formalization of Chain Complexes



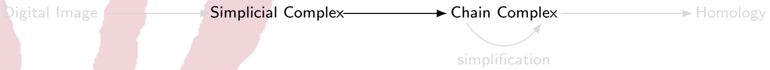
- Formalized in ACL2, Isabelle and Coq

L. Lambán, F. J. Martín-Mateos, J. L. Ruiz-Reina and J. Rubio. When first order is enough: the case of Simplicial Topology. Preprint.

J. Heras and V. Pascual. An ACL2 infrastructure to formalize Kenzo Higher-Order constructors. Preprint.

J. Aransay and C. Domínguez. Modelling Differential Structures in Proof Assistants: The Graded Case. In Proceedings 12th International Conference on Computer Aided Systems Theory (EUROCAST'2009), volume 5717 of Lecture Notes in Computer Science, pages 203–210, 2009.

From Simplicial Complexes to Chain Complexes

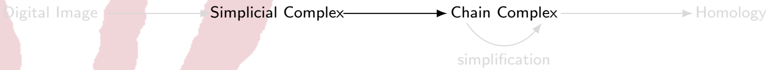


Simplicial Complexes → Simplicial Sets

- Formalized in ACL2

J. Heras, V. Pascual and J. Rubio, Proving with ACL2 the correctness of simplicial sets in the Kenzo system. In LOPSTR 2010, Lecture Notes in Computer Science. Springer-Verlag.

From Simplicial Complexes to Chain Complexes



Simplicial Complexes → Simplicial Sets → Chain Complexes

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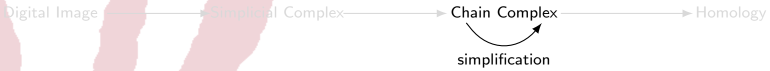
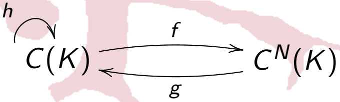
Simplification: Reductions





- Formalized in ACL2

  L. Lambán, F. J. Martín-Mateos, J. L. Ruiz-Reina and J. Rubio. When first order is enough: the case of Simplicial Topology. Preprint.

Simplification: Perturbation techniques

Digital Image → Simplicial Complex → Chain Complex → Homology

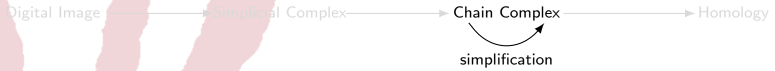
simplification

- EPL:
  - Formalized in ACL2, Coq and Isabelle

J. Heras and V. Pascual. An ACL2 infrastructure to formalize Kenzo Higher-Order constructors. Preprint.

J. Aransay and C. Domínguez. Modelling Differential Structures in Proof Assistants: The Graded Case. In Proceedings 12th International Conference on Computer Aided Systems Theory (EUROCAST'2009), volume 5717 of Lecture Notes in Computer Science, pages 203–210, 2009.

Simplification: Perturbation techniques



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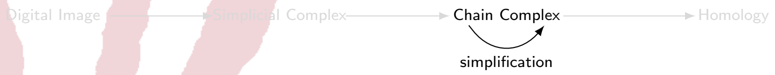
- BPL:
  - Formalized in Isabelle/HOL

    J. Aransay, C. Ballarin and J. Rubio. A mechanized proof of the Basic Perturbation Lemma. Journal of Automated Reasoning, 40(4):271–292, 2008.

  - Formalization of Bicomplexes in Coq

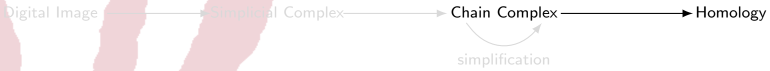
    C. Domínguez and J. Rubio. Effective Homology of Bicomplexes, formalized in Coq. To appear in Theoretical Computer Science.

Simplification: Discrete Morse Theory



- Formalization of Discrete Morse Theory:
  - Work in progress

Homology Groups



- Formalization of Discrete Morse Theory:
  - Work in progress
- Formalization of Homology groups:
  - Future Work

Formalization of digital images

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification

- 2D digital images:
  - Binary matrix

- Formalized in Coq:

R. O'Connor. A Computer Verified Theory of Compact Sets. In SCSS 2008, RISC Linz Report Series.

From Digital Images to Simplicial Complexes

Digital Image → Simplicial Complex → Chain Complex → Homology

simplification

- Elements of digital images $\rightsquigarrow$ Facets of a Simplicial Complex
  - Future work

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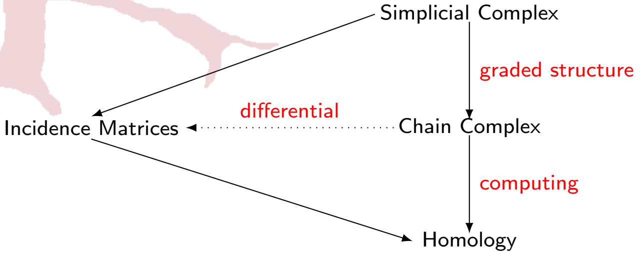
From Simplicial Complexes to Homology



SSReflect

- SSReflect:
  - Extension of Coq
  - Developed while formalizing the Four Color Theorem
  - Provides new libraries:

SSReflect

- SSReflect:
  - Extension of Coq
  - Developed while formalizing the Four Color Theorem
  - Provides new libraries:
    - matrix.v: matrix theory
    - finset.v and fintype.v: finite set theory and finite types
    - bigops.v: indexed "big" operations, like $\sum_{i=0}^{n} f(i)$ or $\bigcup_{i \in I} f(i)$
    - zmodp.v: additive group and ring $\mathbb{Z}_p$

Representation of Simplicial Complexes in SSReflect

Definition

Let V be a finite ordered set, called the vertex set, a simplex over V is any finite subset of V.

```
Variable V : finType.

Definition simplex := {set V}.
```

Representation of Simplicial Complexes in SSReflect

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Let V be a finite ordered set, called the vertex set, a simplex over V is any finite subset of V.

Definition

A finite ordered (abstract) simplicial complex over V is a finite set of simplices $\mathcal{K}$ over V satisfying the property:

$$\forall \alpha \in \mathcal{K},\ \textit{if}\ \beta \subseteq \alpha \Rightarrow \beta \in \mathcal{K}$$

```
Variable V : finType.

Definition simplex := {set V}.

Definition good_sc (c : {set simplex}) :=
    forall x, x \in c -> forall y : simplex, y \subset x ->  y \in c.
```

Incidence Matrices

Definition

Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix m × n where

$$m = \sharp|X| \ \wedge \ n = \sharp|Y|$$

$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} \end{array}$$

$$a_{i,j} = \begin{cases} 1 & \textit{if } X[i] \textit{ is a face of } Y[j] \\ 0 & \textit{if } X[i] \textit{ is not a face of } Y[j] \end{cases}$$

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```
Lemma lt12 : 1 < 2. Proof. by done. Qed.
Definition Z2_ring := (Zp_ring lt12).
Lemma p : 0 < 2. Proof. by done. Qed.
Definition p_0_2 := inZp p 0.
Definition p_1_2 := inZp p 1.
```

Incidence Matrices

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Let X and Y be two ordered finite sets of simplices, we call incidence matrix to a matrix $m \times n$ where

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$$M = \begin{array}{c} \\ X[1] \\ \vdots \\ X[m] \end{array} \begin{array}{c} \begin{array}{ccc} Y[1] & \cdots & Y[n] \end{array} \\ \left(\begin{array}{ccc} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{array}\right) \end{array}$$

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```
Variables Top Left:{set simplex}.

Definition seq_SS (SS: {set simplex}):= enum (mem SS) : seq simplex.
```

Incidence Matrices

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```
Definition incidenceFunction (i j : nat) :=
    if (nth x0 (seq_SS Left) i) \subset (nth x0 (seq_SS Top) j) then p_1_2 else p_0_2.

Definition incidenceMatrix := matrix_of_fun incidenceFunction
                           (m:=size (seq_SS Left)) (n:=size (seq_SS Top)) (R:=Z2_ring).
```

Incidence Matrices

Definition

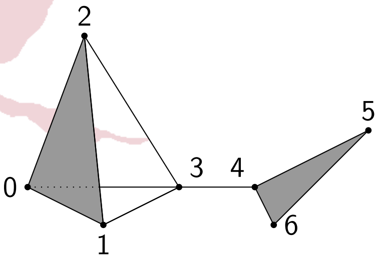
Let C be a finite set of simplices, A be the set of n-simplices of C with an order between its elements and B the set of $(n-1)$-simplices of C with an order between its elements.
We call incidence matrix of dimension n $(n \geq 1)$, to a matrix $p \times q$ where

$$p = \sharp|B| \ \wedge \ q = \sharp|A|$$

$$M_{i,j} \quad = \quad \begin{cases} 1 & \textit{if } B[i] \textit{ is a face of } A[j] \\ 0 & \textit{if } B[i] \textit{ is not a face of } A[j] \end{cases}$$

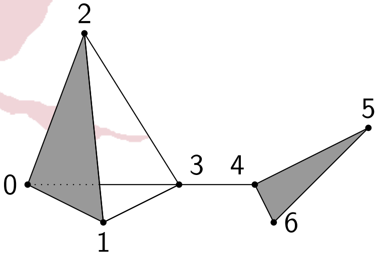
```
Variable c: {set simplex}.

Variable n:nat.

Definition top_n := [set x \in c | #|x| == n+1] : {set simplex}.

Definition left_n_1 := [set x \in c | #|x| == n] : {set simplex}.

Definition incidence_matrix_n := incidenceMatrix top_n left_n_1.
```

Incidence Matrices of Simplicial Complexes



	(0, 1)	(0, 2)	(0, 3)	(1, 2)	(1, 3)	(2, 3)	(3, 4)	(4, 5)	(4, 6)	(5, 6)
(0)	1	1	1	0	0	0	0	0	0	0
(1)	1	0	0	1	1	0	0	0	0	0
(2)	0	1	0	1	0	1	0	0	0	0
(3)	0	0	1	0	1	1	1	0	0	0
(4)	0	0	0	0	0	0	1	1	1	0
(5)	0	0	0	0	0	0	0	1	0	1
(6)	0	0	0	0	0	0	0	0	1	1

Incidence Matrices of Simplicial Complexes



	(0, 1, 2)	(4, 5, 6)
(0, 1)	1	0
(0, 2)	1	0
(0, 3)	0	0
(1, 2)	1	0
(1, 3)	0	0
(2, 3)	0	0
(3, 4)	0	0
(4, 5)	0	1
(4, 6)	0	1
(5, 6)	0	1

Product of two consecutive incidence matrices in $\mathbb{Z}_2$

Theorem (Product of two consecutive incidence matrices in $\mathbb{Z}_2$)

Let $\mathcal{K}$ be a finite simplicial complex over V with an order between the simplices of the same dimension and let $n \geq 1$ be a natural number n, then the product of the n-th incidence matrix of $\mathcal{K}$ and the $(n+1)$-incidence matrix of $\mathcal{K}$ over the ring $\mathbb{Z}/2\mathbb{Z}$ is equal to the null matrix.

```
Theorem incidence_matrices_sc_product:
  forall (V:finType)(n:nat)(sc: {set (simplex V)}), good_sc  sc -> n >= 1 ->
     mulmx (R:=Z2_ring) (incidence_matrix_n  sc n) (incidence_matrix_n  sc (n+1)) =
     null_mx Z2_ring (size (seq_SS  (left_n_1  sc n))) (size (seq_SS (top_n  sc (n+1)))).
```

Sketch of the proof

- Let S_{n+1} be the set of $(n+1)$-simplices of $\mathcal{K}$ with an order between its elements
- Let S_n be the set of n-simplices of $\mathcal{K}$ with an order between its elements
- Let S_{n-1} be the set of $(n-1)$-simplices of $\mathcal{K}$ with an order between its elements

Sketch of the proof

- Let S_{n+1} be the set of $(n+1)$-simplices of $\mathcal{K}$ with an order between its elements
- Let S_n be the set of n-simplices of $\mathcal{K}$ with an order between its elements
- Let S_{n-1} be the set of $(n-1)$-simplices of $\mathcal{K}$ with an order between its elements

$$M_n(\mathcal{K}) = \begin{array}{c} S_{n-1}[1] \\ \vdots \\ S_{n-1}[r2] \end{array} \overset{\begin{array}{ccc} S_n[1] & \cdots & S_n[r1] \end{array}}{\begin{pmatrix} a_{1,1} & \cdots & a_{1,r1} \\ \vdots & \ddots & \vdots \\ a_{r2,1} & \cdots & a_{r2,r1} \end{pmatrix}}, \quad M_{n+1}(\mathcal{K}) = \begin{array}{c} S_n[1] \\ \vdots \\ S_n[r1] \end{array} \overset{\begin{array}{ccc} S_{n+1}[1] & \cdots & S_{n+1}[r3] \end{array}}{\begin{pmatrix} b_{1,1} & \cdots & b_{1,r1} \\ \vdots & \ddots & \vdots \\ b_{r1,1} & \cdots & b_{r1,r3} \end{pmatrix}}$$

where $r1 = \sharp|S_n|$, $r2 = \sharp|S_{n-1}|$ and $r3 = \sharp|S_{n+1}|$

Sketch of the proof

$$M_n(\mathcal{K}) \times M_{n+1}(\mathcal{K}) = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,\ j} = \sum_{1 \leq j0 \leq r1} a_{i,\ j0} \times b_{j0,\ j}$$

Sketch of the proof

$$M_n(\mathcal{K}) \times M_{n+1}(\mathcal{K}) = \begin{pmatrix} c_{1,1} & \cdots & c_{1,r3} \\ \vdots & \ddots & \vdots \\ c_{r2,1} & \cdots & c_{r2,r3} \end{pmatrix}$$

where

$$c_{i,\ j} = \sum_{1 \leq j0 \leq r1} a_{i,\ j0} \times b_{j0,\ j}$$

we need to prove that

$$\forall i, j, c_{i,\ j} = 0$$

in order to prove that $M_n \times M_{n+1} = 0$

Sketch of the proof

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} = \sum_{j0 | S_{n-1}[i] \subset S_n[j0] \wedge S_n[j0] \subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} + \sum_{j0 | S_{n-1}[i] \not\subset S_n[j0] \wedge S_n[j0] \subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} + \sum_{j0 | S_{n-1}[i] \subset S_n[j0] \wedge S_n[j0] \not\subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} + \sum_{j0 | S_{n-1}[i] \not\subset S_n[j0] \wedge S_n[j0] \not\subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j}$$

Sketch of the proof

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} = \begin{array}{l} \displaystyle\sum_{j0 | S_{n-1}[i] \subset S_n[j0] \wedge S_n[j0] \subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \\ + \displaystyle\sum_{j0 | S_{n-1}[i] \not\subset S_n[j0] \wedge S_n[j0] \subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \\ + \displaystyle\sum_{j0 | S_{n-1}[i] \subset S_n[j0] \wedge S_n[j0] \not\subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \\ + \displaystyle\sum_{j0 | S_{n-1}[i] \not\subset S_n[j0] \wedge S_n[j0] \not\subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \end{array}$$

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} = \Big(\sum_{j0 | M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} 1 \Big) + 0 + 0 + 0$$

Sketch of the proof

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} \quad = \quad \begin{aligned} & \sum_{j0 | S_{n-1}[i] \subset S_n[j0] \wedge S_n[j0] \subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \\ + & \sum_{j0 | S_{n-1}[i] \not\subset S_n[j0] \wedge S_n[j0] \subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \\ + & \sum_{j0 | S_{n-1}[i] \subset S_n[j0] \wedge S_n[j0] \not\subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \\ + & \sum_{j0 | S_{n-1}[i] \not\subset S_n[j0] \wedge S_n[j0] \not\subset S_{n+1}[j]} a_{i,\ j0} \times b_{j0,\ j} \end{aligned}$$

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} \quad = \quad \left(\sum_{j0 | M_{n-2}[i] \subset M_{n-1}[j0] \wedge M_{n-1}[j0] \subset M_n[j]} 1 \right) + 0 + 0 + 0$$

$$\sum_{1 \leqslant j0 \leqslant r1} a_{i,\ j0} \times b_{j0,\ j} \quad = \quad \sharp |\{ j0 | (1 \leq j0 \leq r1) \wedge (S_{n-1}[i] \subset S_n[j0]) \wedge (S_n[j0] \subset S_{n+1}[j]) \}|$$

Sketch of the proof

$$\sharp|\{j0 \mid (1 \leq j0 \leq r1) \wedge (S_{n-1}[i] \subset S_n[j0]) \wedge (S_n[j0] \subset S_{n+1}[j])\}| = 0 \bmod 2$$

Sketch of the proof

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- $S_{n-1}[i] \not\subset S_{n+1}[j] \Rightarrow$

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Otherwise, $\exists k$ such that $S_{n-1}[i] \subset S_n[k] \subset S_{n+1}[j]$

Sketch of the proof

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$$S_{n+1}[j] = (v_0, \ldots, v_{n+1})$$

$$S_{n-1}[i] = (v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{n+1})$$

Sketch of the proof

$$\sharp|\{j0 \mid (1 \leq j0 \leq r1) \wedge (S_{n-1}[i] \subset S_n[j0]) \wedge (S_n[j0] \subset S_{n+1}[j])\}| = 0 \bmod 2$$

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$$S_{n+1}[j] = (v_0, \ldots, v_{n+1})$$

$$S_n[k_1] = (v_0, \ldots, \widehat{v_j}, \ldots, v_{n+1}) \nearrow \nwarrow \qquad S_n[k_2] = (v_0, \ldots, \widehat{v_i}, \ldots, v_{n+1}) \nwarrow \nearrow$$

$$S_{n-1}[i] = (v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{n+1})$$

$\square$

Formalization in SSReflect

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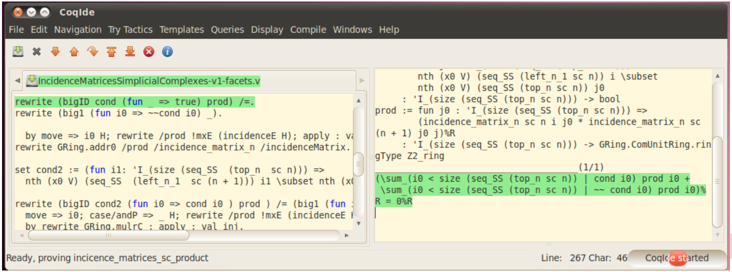
  - `bigID`: $\sum_{i \in r | P_i} F_i = \sum_{i \in r | P_i \wedge a_i} F_i + \sum_{i \in r | P_i \wedge \sim a_i} F_i$
  - `big1`: $\sum_{i \in r | P_i} 0 = 0$



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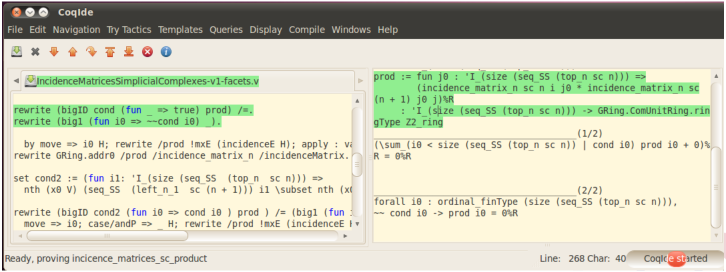

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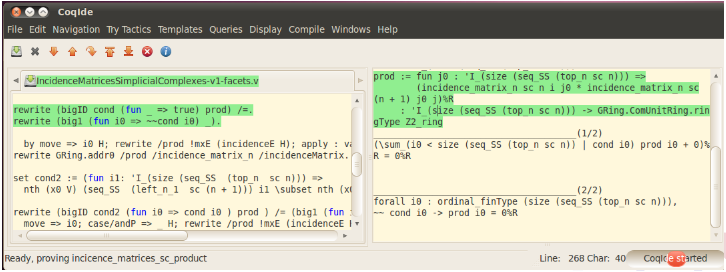

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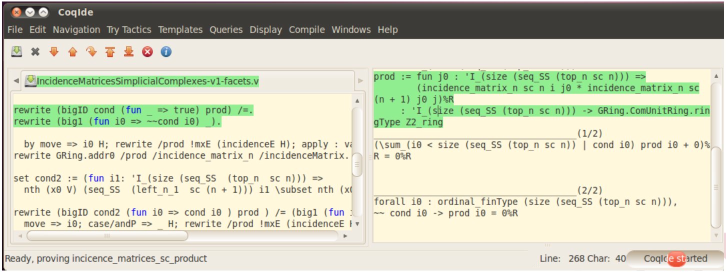
  - big1: $\sum_{i \in r | P_i} 0 = 0$
- Cardinality part: More details are needed
  - Auxiliary lemmas
  - Lemmas from “finset” and “fintype” libraries



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Conclusions and Further Work

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- Future Work:
  - Formalization: Digital Images to Simplicial Complexes
  - Formalization of Discrete Morse Theory
  - Formalization of computation of Homology groups
  - ...

Formal libraries for Algebraic Topology: status report

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