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VALIDITY OF MANY-BODY APPROXIMATION METHODS FOR A SOLVABLE MODEL

(I). Exact Solutions and Perturbation Theory

H. J. LIPKIN,
Weizmann Institute of Science, Rehovoth, Israel
N. MESHKOV and A. J. GLICK †
Weizmann Institute of Science, Rehovoth, Israel
and
University of Maryland, College Park, Maryland ††

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Abstract: In order to test the validity of various techniques and formalisms developed for treating many-particle systems, a model is constructed which is simple enough to be solved exactly in some cases, but yet is non-trivial. The construction of such models is based on the observation that bilinear products of creation and annihilation operators can be considered as generators of Lie groups. Thus the problem of finding eigenvalues can be greatly simplified by the additional integrals of the motion which are present if the Hamiltonian is constructed so as to commute with invariants of the group. In the present case, the model consists of N fermions distributed in two N-fold degenerate levels and interacting via a monopole-monopole force. It is shown that the model Hamiltonian is easily expressed in terms of quasi-spin operators and exact eigenvalues are obtained. In addition, eigenvalues are calculated with ordinary perturbation theory using values for the number of particles and interaction strength which are appropriate to the more realistic problems of finite nuclei. In subsequent papers we consider the results obtained by various other approximation methods for comparison with the exact results presented here.

1. Introduction

Recently many techniques and formalisms have been developed for treating many-particle systems. Although different in appearance, some of these methods seem to be simply related if not completely equivalent. However, direct comparison of treatments using different complicated notations is difficult. It is also difficult to check the validity of various approximations as a function of the characteristic parameters occurring in the physical systems. In particular, one can question the application to nuclear physics of approximation methods such as the random phase approximation and the BCS treatment of pairing correlations [1, 2]), which have been developed in field theory and solid state physics where the number of degrees of freedom in the system is very large.

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These different formulations can be tested in models which are simple enough to be solved exactly but are yet non-trivial. By comparing the approximate solutions with the exact ones, one can determine the region of validity for each method and perhaps also find modifications which will have a wider range of validity.

One method for constructing soluble models is to incorporate some symmetry in the system which gives additional integrals of the motion and therefore can drastically reduce the size of the Hamiltonian matrix to be diagonalized. Symmetries can be introduced into a Hamiltonian written in terms of creation and annihilation operators by noting that bilinear products of these operators can be considered as generators of Lie groups. The commutator of two bilinear products of creation and annihilation operators is a linear combination of such bilinear products. Therefore, the set of all possible bilinear products formed from a finite set of these operators constitutes a Lie algebra. The Hamiltonian for a many-particle system interacting via two-body forces is a sum of linear and quadratic terms in these bilinear products and is thus expressible as a function of the operators of a Lie algebra or the generators of a Lie group. The invariants or Casimir operators of the group then commute with the Hamiltonian and provide constants of the motion in addition to those provided by the usual conservation laws.

The Elliott model [3]) for light nuclei is an example of a soluble model whose Hamiltonian is expressible in terms of the generators of a Lie group; in this case the group SU_3. The simplest Lie algebra is the angular momentum algebra. There are a number of ways of constructing bilinear products of creation and annihilation operators which satisfy commutation rules like angular momenta. Such operators are often called quasi-spins because of the formal resemblance to angular momentum.

In this paper, we consider a model which can be described in terms of quasi-spins and for which exact solutions are obtainable in many cases. The model also possesses states of collective excitation in nuclear physics similar to the giant resonances observed and often treated using the random phase approximation [4]). This model has been previously used by Fallieros [5]) as a simplified model of a giant monopole resonance. The purpose of this paper is to show that the model Hamiltonian is easily expressed in terms of quasi-spin operators, and to give exact solutions as well as solutions with ordinary perturbation theory using values for the number of particles and interaction strength which are relevant to the treatment of nuclei by the random phase approximation. In subsequent papers we consider the result obtained by various formulations of the random phase approximation and compare them with these exact results.

2. The Model

Consider a system of N fermions distributed in two levels each having an N-fold degeneracy and separated by an energy ε. These levels could be two jj-coupling shells with the same j. Each state is described by a quantum number σ which has the value $+1$ in the upper shell and -1 in the lower shell, and a quantum number p specifying

the particular degenerate state within the shell. For each value of p there are two corresponding states, one in the lower shell and one in the upper shell. If the levels are two jj coupling shells, the quantum number p is the quantum number m specifying the z-component of the angular momentum. A two-body interaction is assumed which scatters pairs of particles between the two levels without changing the value of p. In the case when the levels are jj-coupling shells, such an interaction would be a "monopole-monopole" interaction in which the angular momentum quantum numbers of the particle are not changed in the scattering. Another interpretation of this model could be two harmonic oscillator major shells of the same parity containing several jj-coupling sub shells with a monopole interaction scattering particles between the shells without changing the angular momentum quantum numbers.

Let $a^\dagger_{p\sigma}$ be the creation operator for a particle in the p state of the σ level. The Hamiltonian for the system can then be written as

$$H = \tfrac{1}{2}\varepsilon \sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma} + \tfrac{1}{2}V \sum_{pp'\sigma} a^\dagger_{p\sigma} a^\dagger_{p'\sigma} a_{p'-\sigma} a_{p-\sigma} + \tfrac{1}{2}W \sum_{pp'\sigma} a^\dagger_{p\sigma} a^\dagger_{p'-\sigma} a_{p'\sigma} a_{p-\sigma}, \quad (2.1)$$

where V and W are parameters specifying the strengths of the interactions. The term proportional to V scatters a pair of particles in the same level to the other level. The term proportional to W scatters one particle up while another is scattered down.

The unperturbed ground state for this system (if $V = W = 0$) has all the particles in the lower level, each particle having a different value of the quantum number p. The interaction does not change the value of p and only mixes the unperturbed ground state with those states in which each particle has a different value of the quantum number p but may be either in the upper or lower state. There are thus a total of 2^N states of this kind. The diagonalization of the Hamiltonian (2.1) thus involves a matrix of order 2^N. The symmetry of the problem allows a considerable reduction in the size of the matrix to be diagonalized.

That each particle has only two possible states immediately suggests the use of a quasi-spin formulation. The two-valued specification of the state of a given particle can be represented by a Pauli spinor, while operators describing a transition of a particle from one state to another can be represented by Pauli spin matrices. The total quasi-spin operators of the system would then be defined by the relations

$$J_+ = \sum_p a^\dagger_{p+1} a_{p-1}, \qquad J_- = \sum_p a^\dagger_{p-1} a_{p+1}, \qquad J_z = \tfrac{1}{2} \sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma}. \quad (2.2)$$

It can be seen by inspection that the operators (2.2) satisfy angular momentum commutation rules. The Hamiltonian is easily expressed in terms of these operators as

$$H = \varepsilon J_z + \tfrac{1}{2}V(J_+^2 + J_-^2) + \tfrac{1}{2}W(J_+ J_- + J_- J_+). \quad (2.3)$$

The operator $J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2$ commutes with the Hamiltonian (2.3). Thus the Hamiltonian matrix breaks up into submatrices, each associated with a different value of J and of order $2J+1$. The operator J_z is just half the difference between the number of particles in the upper state and the number of particles in the lower state.

Thus the maximum possible value for J_z and therefore for J is $\frac{1}{2}N$. The symmetry † has therefore reduced the size of the largest matrix to be diagonalized from 2^N to $N+1$.

Let us now consider the properties of the multiplet of states belonging to a particular submatrix with an eigenvalue J. The operator J_z can also be written as

$$J_z = n_{\mathrm{ph}} - \tfrac{1}{2}N,$$

where n_{ph} is the number of excited particle-hole pairs. Since in a given multiplet each value of J_z occurs only once, there is only one state having a given number of excited particle-hole pairs in each multiplet.

The multiplet containing the unperturbed ground state has the maximum value of J, namely, $J = \frac{1}{2}N$, since the unperturbed ground state has $J_z = -\frac{1}{2}N$. The multiplet contains one state with one excited particle-hole pair, one state with two excited particle-hole pairs, etc., etc. The interaction mixes the states within the same J multiplet. To see the significance of the quasi-spin symmetry, consider the case of a typical double-closed-shell nucleus such as O^{16} as treated in the shell model. The first step (e.g., the classical treatment by Elliott and Flowers [3]) considers all possible states formed by the excitation of a single particle-hole pair from the ground state and diagonalizes the interaction in this subspace. In diagrammatic language this is called summing all the forward-going graphs. The next step, sometimes called the inclusion of ground-state correlations, consists in taking into account in some fashion the contribution of forward-and-backward-going graphs which correspond to the mixing into the ground state of states having several excited particle-hole pairs, and into the excited state of states with more than one excited particle-hole pair. In our model, the quasi-spin symmetry automatically performs the standard shell model diagonalization or summation of forward-going graphs. If we are considering, for example, the states involving one excited particle-hole pair, these are distributed among multiplets in such a way that there is only *one* state with one excited particle-hole pair in each multipet. The residual interaction does not mix these one-particle-one-hole states with one another. It merely mixes each one-particle-one-hole state with the other members of the same multiplet containing different numbers of excited particle-hole pairs. The shell model diagonalization is therefore trivial in this model which is of primary interest for the study of the treatment of ground-state correlations.

The structure of the states arising in this model can be examined further as follows. We consider first the unperturbed system by setting the interaction $V = W = 0$. We then find that the ground state has $J_z = -J = -\frac{1}{2}N$. The first-excited state is N-fold degenerate since there are N possible ways to excite one particle-hole pair. These states all have $J_z = -\frac{1}{2}N+1$, and are mixtures of eigenfunctions of the operator J^2 with eigenvalues $-J$ and $-J+1$. One particular linear combination of these

† Another description of the quasi-spin symmetry is in terms of the invariance of the Hamiltonian (2.1) under permutations of the different p states. The different values of J^2 correspond to different representations of the permutation group. The ground and collective excited states are all totally symmetric under these permutations.

one-particle-one-hole states is an eigenfunction of J^2 with eigenvalue $-J$. This state is the one obtained by operating on the ground state with the operator J_+. All other one-particle-one-hole states orthogonal to this one must have $J = \frac{1}{2}N-1$.

We now let the interaction parameter W become finite but keep $V = 0$, to remove some of the degeneracy without mixing states having different numbers of particle-hole pairs. This corresponds to the case of the conventional shell-model. The first-excited state is now split into two energy levels, one for each value of J. The single state having $J = \frac{1}{2}N$ splits off from the rest and can be considered as a "collective" state since it is generated from the ground state by the action of the "collective" operator J_+. If one defines a "monopole operator" as some linear combination of J_+ and J_-, this collective state absorbs the entire monopole sum rule since it is just the state generated from the ground state by the monopole operator. The remaining $N-1$ states with one excited particle-hole pair can be considered as "single-particle" states. All these single-particle excitations are degenerate in this model, even when the interaction is large.

We now consider the case when the interaction V is finite and produces configuration mixing, i.e., mixes states having different numbers of excited particle-hole pairs. Since this cannot mix states having different eigenvalues of J^2, the particular one-particle-one-hole state which we have designated as collective cannot be mixed with other one-particle one-hole states but is mixed with states having different numbers of particles and holes. Another feature of the mixing results from the particular form of the interaction which can only excite or de-excite two-particle-hole pairs. States having even and odd numbers of particle-hole pairs are therefore not mixed by this interaction. This property is expressed in the quasi-spin language by noting that the interaction can only change the eigenvalue of J_z by two units. Thus, for integral values of J, the states having even and odd eigenvalues of J_z are not mixed. For half-integral eigenvalues of J (odd numbers of particles) the states with eigenvalues $\ldots -\frac{3}{2}, \frac{1}{2}, \frac{5}{2} \ldots$ do not mix with the states having eigenvalues $-\frac{5}{2}, -\frac{1}{2}, \frac{3}{2} \ldots$ That the interaction cannot excite a single-particle-hole pair indicates that the particular single-particle basis chosen for the model satisfies the Hartree-Fock equations for the interaction.

The interaction term proportional to W does not mix configurations and is diagonalized exactly by the quasi-spin representation. The interaction term proportional to V contributes the ground-state correlations and mixes configurations. Since the main purpose of this model is to test the treatment of ground-state correlations in the random phase approximation, the Hamiltonian (2.3) will be simplified by setting $W = 0$. The resulting Hamiltonian

$$H = \varepsilon J_z + \tfrac{1}{2}V(J_+^2 + J_-^2) = \varepsilon J_z + V(J_x^2 - J_y^2) \tag{2.4}$$

now has the following additional symmetry. A rotation of 180° in quasi-spin space about an axis in the xy plane at an angle of 45° to the x and y axes changes H into $-H$. Such a transformation on any eigenfunction of H with eigenvalue E produces

another eigenfunction of H with eigenvalue $-E$. The eigenvalues of H therefore occur in pairs and the secular equation for the eigenvalues contains only even or only odd powers of E depending upon the size of the matrix.

Another possible choice for the interaction parameters is $W = V$. This leads to a Hamiltonian having a particularly simple form in the strong coupling limit:

$$H_s = \varepsilon J_z + 2VJ_x^2. \tag{2.5}$$

The Hamiltonian (2.5) is easily solved exactly in the strong coupling limit $V \gg \varepsilon$ where the exact eigenfunctions are eigenfunctions of J^2 and J_x. However, the Hamiltonian (2.5) does not possess the 180° rotation symmetry described above for the Hamiltonian (2.4). The Hamiltonian (2.5) might be useful for the study of the "instability of the Hartree-Fock state against collective oscillations" where the random phase approximation breaks down. One can then find new Hartree-Fock solutions which are eigenfunctions neither of J_x nor of J_z but of some linear combination describing an orientation somewhere in the xz plane. Detailed calculations of Hartree-Fock instability and the strong coupling limit are not considered further in this paper and the remainder of this work is concerned with the properties of the Hamiltonian (2.4) for interaction strengths where the random phase approximation can still be expected to be valid; i.e., the Hartree-Fock state is stable against the collective oscillations.

3. Exact Solutions

The exact eigenfunctions of Hamiltonian (2.4) are eigenfunctions of J^2 and linear combinations of the $2J+1$ corresponding eigenfunctions of J_z. For a system containing a given number of particles N, the Hamiltonian matrix breaks up into submatrices of which the largest corresponds to $J = \frac{1}{2}N$ and includes the ground state and the collective excited states of interest to this paper. We shall therefore consider only those states in the particular multiplet with $J = \frac{1}{2}N$. Note, however, that the Hamiltonian (2.5) depends only upon the quasi-spin variables and not explicitly upon the number of particles. A solution for a given value of J thus applies not only to the ground state multiplet for $N = 2J$ but also to all multiplets for larger numbers of particles having the same value of J.

The two additional symmetries further simplify the diagonalization. The $2J+1$ states are split into two groups, each containing eigenfunctions of J_z with eigenvalues differing by an even number. The degree of each secular equation is further reduced by a factor of 2 because the energy eigenvalues come in equal and opposite pairs. The diagonalization of the matrix for N particles thus requires solution of algebraic equations which are of degree $\frac{1}{2}(N+1)$ if N is odd, $\frac{1}{4}(N+2)$, if N is twice an odd number, and $\frac{1}{4}N$ if N is a multiple of 4. In particular for $N = 8$, $J = 4$ and the 9×9 matrix splits up into a 5×5 one for even eigenvalues of J_z and a 4×4 one with odd values of J_z. The secular equation for the 5×5 matrix is a fifth degree one and has one zero eigenvalue and two pairs of equal and opposite eigenvalues.

The energy levels of the system are characterized by two parameters, the number of particles N and the interaction strength V/ε. Since for $N = 2, 3, 4, 6, 8$ the secular equation is at most quadratic, exact solutions of eq. (2.4) can be obtained analytically for these values of N. The $N+1$ eigenvalues for the given N are

for $N = 2$:

$$\frac{E}{\varepsilon} = \pm\left[1+\left(\frac{V}{\varepsilon}\right)^2\right]^{\frac{1}{2}},\ 0, \tag{3.1}$$

for $N = 3$:

$$\frac{E}{\varepsilon} = \pm\left\{\tfrac{1}{2}\pm\left[1+3\left(\frac{V}{\varepsilon}\right)^2\right]^{\frac{1}{2}}\right\}, \tag{3.2}$$

for $N = 4$:

$$\begin{aligned}\frac{E}{\varepsilon} &= 0,\ \pm 2\left[1+3\left(\frac{V}{\varepsilon}\right)^2\right]^{\frac{1}{2}},\\ \frac{E}{\varepsilon} &= \pm\left[1+9\left(\frac{V}{\varepsilon}\right)^2\right]^{\frac{1}{2}}\end{aligned} \tag{3.3}$$

for $N = 6$:

$$\begin{aligned}\frac{E}{\varepsilon} &= \pm\left\{5+33\left(\frac{V}{\varepsilon}\right)^2 \pm\left[1+6\left(\frac{V}{\varepsilon}\right)^2+54\left(\frac{V}{\varepsilon}\right)^4\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}},\\ \frac{E}{\varepsilon} &= 0,\ \pm 2\left[1+15\left(\frac{V}{\varepsilon}\right)^2\right]^{\frac{1}{2}},\end{aligned} \tag{3.4}$$

for $N = 8$:

$$\begin{aligned}\frac{E}{\varepsilon} &= \pm\left\{5+113\left(\frac{V}{\varepsilon}\right)^2 \pm 4\left[1+38\left(\frac{V}{\varepsilon}\right)^2+550\left(\frac{V}{\varepsilon}\right)^4\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}},\\ \frac{E}{\varepsilon} &= 0,\ \pm\left\{10+118\left(\frac{V}{\varepsilon}\right)^2 \pm\left[1-2\left(\frac{V}{\varepsilon}\right)^2+225\left(\frac{V}{\varepsilon}\right)^4\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

In order to illustrate the behaviour of eigenvalues as a function of the interaction parameter NV/ε, the positive solutions for $N = 8$ are plotted in fig. 1. In addition, for $N = 14$, 30 and 50, eigenvalues were found numerically for several values of the interaction parameter NV/ε. The results of numerical calculations are presented in tables 1–3.

In what follows, we shall be primarily concerned with the excitation energy of the first-excited state above the ground state. The exact results for the excitation energy as a function of the interaction parameter are shown in figs. 2 and 3. Fig. 3 also contains the results of second-order perturbation theory discussed in the following section.

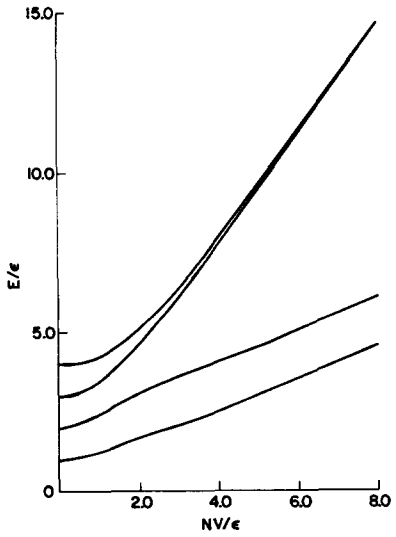

Fig. 1. Positive energy eigenvalues plotted versus the interaction parameter NV/ε for 8 particles.

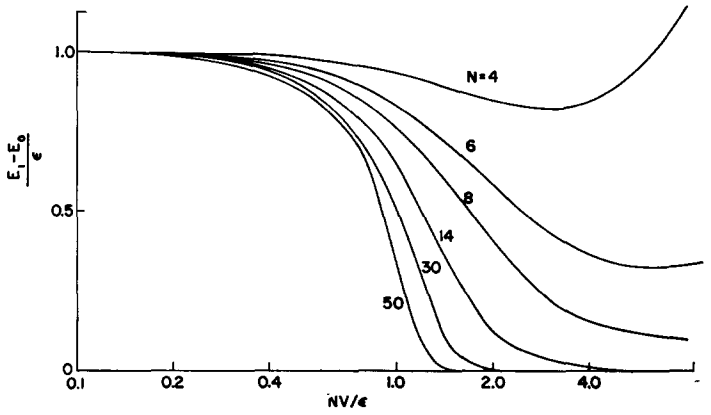

Fig. 2. Exact results for the excitation energy of the first excited state above the ground state plotted versus the interaction parameter NV/ε for $N = 4$, 6, 8, 14, 30 and 50 particles.

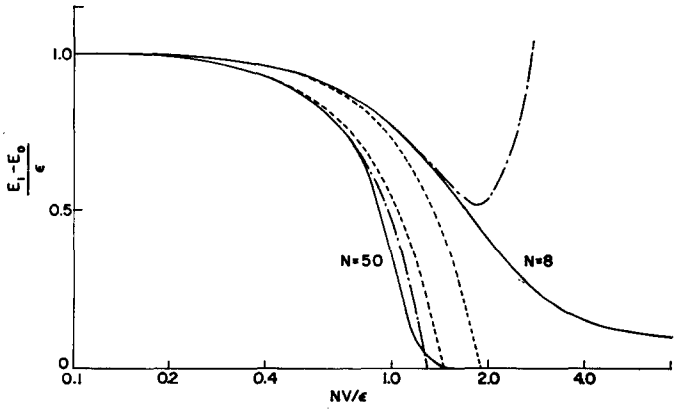

Fig. 3. Excitation energy of the first-excited state above the ground state. The upper three lines refer to the eight particle system and the lower three lines to $N = 50$. The solid lines shows the exact excitation energy, the short dashed lines represent the second-order perturbation theory results, and the dot-and-dash lines show the results of perturbation theory to fourth order.

TABLE 1

Positive energy eigenvalues E/ε of the model Hamiltonian (2.4) for 14 particles for different values of the interaction parameters NV/ε

$NV/\varepsilon = 0$	0.4	0.6	0.8	1.0	2.0	5.0	∞
7.000	7.038	7.088	7.163	7.270	8.636	17.268	46.151
6.000	6.096	6.219	6.393	6.622	8.517	17.264	46.151
5.000	5.123	5.273	5.475	5.717	6.983	11.185	35.396
4.000	4.126	4.274	4.467	4.692	6.025	10.984	28.000
3.000	3.110	3.236	3.398	3.583	4.639	7.070	28.028
2.000	2.080	2.172	2.287	2.418	3.164	5.492	12.889
1.000	1.042	1.090	1.150	1.218	1.601	2.671	13.416
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The last column gives the ratio E/V in the strong coupling limit ($NV/\varepsilon \to \infty$).

TABLE 2

Positive energy eigenvalues E/ε of the model Hamiltonian (2.4) for 30 particles for different values of the interaction parameter NV/ε

$NV/\varepsilon=0$	0.4	0.6	0.8	1.0	2.0	5.0	∞
15.000	15.040	15.094	15.179	15.314	18.547	38.049	218.83
14.000	14.111	14.258	14.480	14.800	18.545	38.049	218.83
13.000	13.165	13.378	13.684	14.087	16.453	31.436	178.02
12.000	12.204	12.461	12.819	13.269	16.327	31.436	178.02
11.000	11.230	11.514	11.899	12.371	15.029	25.422	140.28
10.000	10.245	10.540	10.935	11.407	14.242	25.419	140.28
9.000	9.248	9.544	9.933	10.392	13.093	20.142	105.69
8.000	8.243	8.529	8.900	9.334	11.875	20.053	105.69
7.000	7.229	7.497	7.841	8.239	10.557	16.135	74.386
6.000	6.209	6.451	6.759	7.114	9.169	15.244	74.377
5.000	5.183	5.393	5.659	5.964	7.723	12.633	65.449
4.000	4.152	4.326	4.545	4.795	6.231	10.466	46.496
3.000	3.117	3.251	3.418	3.609	4.703	7.933	46.642
2.000	2.080	2.170	2.284	2.413	3.150	5.348	22.035
1.000	1.040	1.086	1.143	1.208	1.579	2.688	23.400
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The last column gives the ratio E/V in the strong coupling limit ($NV/\varepsilon \to \infty$).

TABLE 3

Positive energy eigenvalues E/ε of the model Hamiltoninan (2.4) for 50 particles for different values of the interaction parameter NV/ε

$NV/\varepsilon=0$	0.4	0.6	0.8	1.0	2.0	5.0	∞
25.000	25.041	25.096	25.186	25.340	31.037	64.043	614.70
24.000	24.116	24.273	24.520	24.901	31.037	64.043	614.70
23.000	23.182	23.422	23.785	24.295	28.742	57.300	545.58
22.000	22.237	22.545	22.996	23.599	28.740	57.300	545.58
21.000	21.283	21.645	22.162	22.831	26.762	50.898	479.50
20.000	20.321	20.725	21.291	22.005	26.697	50.898	479.50
19.000	19.351	19.786	20.386	21.130	25.366	44.854	416.50
18.000	18.373	18.831	19.453	20.212	24.821	44.854	416.50
17.000	17.388	17.860	18.493	19.256	23.809	39.193	356.62
16.000	16.397	16.875	17.509	18.267	22.794	39.193	356.62
15.000	15.400	15.877	16.505	17.249	21.668	33.967	299.90
14.000	14.397	14.867	15.482	16.205	20.477	33.960	299.90
13.000	13.389	13.847	14.441	15.136	19.224	29.333	246.42
12.000	12.376	12.817	13.385	14.047	17.920	29.205	246.42
11.000	11.360	11.778	12.316	12.938	16.570	25.786	196.29
10.000	10.339	10.732	11.233	11.813	15.180	24.826	196.29
9.000	9.315	9.678	10.140	10.672	13.756	22.442	149.64
8.000	8.288	8.618	9.037	9.519	12.300	20.400	149.63
7.000	7.258	7.553	7.926	8.353	10.818	18.047	106.70
6.000	6.225	6.483	6.807	7.178	9.314	15.635	106.66
5.000	5.191	5.408	5.682	5.994	7.790	13.134	67.978
4.000	4.155	4.331	4.552	4.804	6.250	10.574	67.620
3.000	3.117	3.251	3.417	3.608	4.699	7.969	34.935
2.000	2.079	2.168	2.280	2.407	3.138	5.330	32.462
1.000	1.040	1.084	1.140	1.204	1.570	2.670	10.057
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

The last column gives the ratio E/V in the strong coupling limit ($NV/\varepsilon \to \infty$).

4. Perturbation Theory

The first approximation procedure employed in the study of this system is ordinary perturbation theory. The energies of the ground and the first-excited states were calculated to second and fourth order in the interaction between particles. The application of perturbation theory to this problem is greatly facilitated by using the quasi-spin representation of Hamiltonian (2.4). A straightforward application of perturbation theory to H in the representation of eq. (2.1) can also be carried out and Wick's theorem used for evaluating the relevant matrix elements of products of annihilation and creation operators. However, this procedure is very lengthy and cumbersome, though the final answer must of course be the same if the perturbation series is at all meaningful.

Using the Hamiltonian (2.4) the second-order correction to the energy of the state with a given J ($J = \frac{1}{2}N$) and m is

$$\Delta^{(2)}_{J,m} = m[J(J+1)-m^2-\tfrac{1}{2}]\left(\frac{V}{\varepsilon}\right)^2, \tag{4.1}$$

where m is the eigenvalue of J_z.

In fourth order this correction is

$$\Delta^{(4)}_{J,m} = \tfrac{1}{2}m[-18(J^2-m^2)^2+2(J^2-m^2)(4J^2-14J^2+27) + (2J-1)(4J^2-14J+9)]\left(\frac{V}{\varepsilon}\right)^4. \tag{4.2}$$

The resulting excitation energy above the ground state is given by

$$\frac{E_1-E_0}{\varepsilon} = 1-\tfrac{1}{2}(N-3)(N-1)\left(\frac{V}{\varepsilon}\right)^2-\tfrac{1}{8}(N-3)(N-1)(N^2-16N+27)\left(\frac{V}{\varepsilon}\right)^4, \tag{4.3}$$

where the three terms correspond to the zero, second and fourth order, respectively.

In order to illlustrate the range of validity of perturbation theory, the results to second and fourth order are plotted in fig. 3 together with the exact excitation energy for 8 and 50 particles.

In the following papers, other approximation methods will be employed for calculating the excitation energy in order to study the regions of validity of these methods by comparing their accuracy with the exact solutions and the perturbation theory results found above. Paper II deals with the method of linearizing the equations of motion and with several improvements on that method. Paper III deals with diagrammatic many-body perturbation theory and various selective summations of higher order graphs.

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